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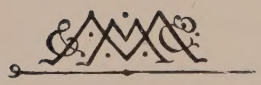
LECTURES ON THEORETICAL PHYSICS



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LECTURES ON THEORETICAL PHYSICS

DELIVERED AT THE UNIVERSITY OF LEIDEN

BY

H. A. LORENTZ

AUTHORISED TRANSLATION

BY

L. SILBERSTEIN, PH.D., AND A. P. H. TRIVELLI

VOLUME I

AETHER THEORIES AND AETHER MODELS

EDITED BY H. BREMEKAMP, PH.D.

KINETICAL PROBLEMS

EDITED BY E. D. BRUINS, PH.D., AND J. REUDLER, PH.D.

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PREFACE TO THE ENGLISH TRANSLATION

A SERIES of admirably clear and instructive courses of lectures, covering essentially almost the whole field of Theoretical Physics, were delivered by Professor Lorentz at the University of Leiden. Seven of these courses, bearing on subjects of fundamental importance, were, up to 1922, edited by Lorentz's pupils and published in the Dutch language. During Professor Lorentz's visit in the United States in 1922 the plan was conceived to make these courses of lectures accessible to English readers.

After some delay necessitated by circumstances the present Volume I. is being issued. It contains the English version of two of these courses of lectures, viz. *Aether Theories and Aether Models*, edited in the Dutch language by Prof. H. Bremekamp, and *Kinetical Problems*, edited by Dr. E. D. Bruins and Dr. J. Reudler. To preserve the peculiar charm of Lorentz's own style and exposition the translation of these lectures has been made as literal as was compatible with the nature of the English language. Only a very few changes or explanatory additions were made in the text. The latter are placed in square brackets.

Volumes II. and III., now in preparation, will contain the remaining five lecture courses, namely, *Thermodynamics*, *Entropy and Probability*, *Theory of Radiation*, *Theory of Quanta*, and *The Principle of Relativity for Uniform Translations* (Special Theory of Relativity).

Recently, 1925, one more course of lectures, on *Maxwell's Theory*, was brought out at Leiden, and arrangements will be made to include its English version into this publication.

L. S.
A. P. H. T.

ROCHESTER, N.Y.,
July 1926.

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AETHER THEORIES AND AETHER MODELS

(1901–1902)

I

ABERRATION OF LIGHT

THE first question which presents itself when we try to form an idea of the nature of the aether is that concerning its relations to ponderable matter. More especially we will consider the question whether a moving body, such as a planet, does drag the surrounding aether. The theory of the aberration of light may give us in this respect some information.

1. STOKES' THEORY: THE EARTH DRAGS THE SURROUNDING AETHER

Stokes imagines the aether streaming in the neighbourhood of a planet and determines the direction in which according to this hypothesis a star should be seen in the following way. If we consider a wave-front near the planet, we can determine any of its successive positions by the *Huygens* construction, provided we take account of the velocity of the aether at each point of the original wave-front. Since the velocities at different points are different, the wave-front will be slightly tilted, and since the direction in which we see the light source is determined by the normal of the wave when it reaches our eye, this rotation will give us an explanation of the aberration. In order to find the amount of the rotation thus produced, we note first of all that the translational [orbital] velocity p of the earth is 10^{-4} times the light velocity V. Terms of the order $(p/V)^2$ will be neglected.

Let the plane AB (Fig. 1) represent the wave-front at the instant t, and let our z-axis OZ be perpendicular to this plane at a point M. Then, if u_0, v_0, w_0 be the velocity components of

the aether at M, we can write for the velocity components at a point x, y, z_0 of AB chosen in the neighbourhood of M,

$$u = u_0 + \frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y,$$

$$v = v_0 + \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y,$$

$$w = w_0 + \frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y.$$

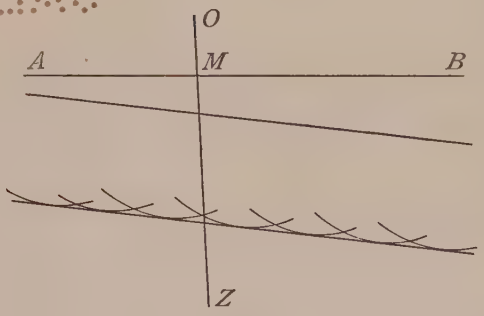

FIG. 1.

During the time dt the point will be displaced to the new position

$$x + \left(u_0 + \frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y\right)dt,\ y + \left(v_0 + \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y\right)dt,$$

$$z_0 + \left(w_0 + \frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y\right)dt.$$

The plane drawn through all the points thus obtained is parallel to the new wave-front. Its equation will be

$$z = z_0 + \left(w_0 + \frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y\right)dt.$$

This relation, in whose second term x and y stand for the old co-ordinates, will hold, provided the differences between these and the new co-ordinates are very small. The direction-cosines of this plane are

$$-\frac{\partial w}{\partial x}dt,\ -\frac{\partial w}{\partial y}dt,\ 1,$$

so that the wave normal makes with the X-axis the angle $\frac{\pi}{2}+\frac{\partial w}{\partial x}dt$, and with the Y-axis the angle $\frac{\pi}{2}+\frac{\partial w}{\partial y}dt$. The changes of the two first direction-cosines, which originally were zero, that is to say, the magnitudes

$$-\frac{\partial w}{\partial x}dt \text{ and } -\frac{\partial w}{\partial y}dt,$$

can be taken as a measure of the rotation of the wave-front about the y- and the x-axis.

The wave-front propagates itself in the time dt over a distance Vdt, so that the rotation, per unit length, has the components

$$-\frac{1}{V}\frac{\partial w}{\partial x}, \quad -\frac{1}{V}\frac{\partial w}{\partial y}.$$

The total rotation undergone by the wave-front until it will have reached our eye will, of course, be expressed by an integral. The preceding reasoning gives us the contribution to that integral for a wave-front which is normal to the adopted Z-axis. The deviations from that orientation remain, however, so small that these results can be used for all the elements of the integral. Thus the components of the total rotation will be

$$-\frac{1}{V}\int\frac{\partial w}{\partial x}dz, \quad -\frac{1}{V}\int\frac{\partial w}{\partial y}dz,$$

where the integrations are to be extended from a point at which the earth's motion is still imperceptible up to the observer's station P.

Now, it is actually possible to account for aberration by means of this rotation of the wave-front, provided that the motion of the aether is assumed to be irrotational. There exists then a velocity potential ϕ, such that

$$u=\frac{\partial\phi}{\partial x}, \quad v=\frac{\partial\phi}{\partial y}, \quad w=\frac{\partial\phi}{\partial z}.$$

Thus also $\partial w/\partial x=\partial u/\partial z$, $\partial w/\partial y=\partial v/\partial z$, and our expressions for the rotation components become

$$-\frac{1}{V}\int\frac{\partial u}{\partial z}dz=-\frac{u_P}{V} \text{ and } -\frac{1}{V}\int\frac{\partial v}{\partial z}dz=-\frac{v_P}{V}.$$

It will be readily seen that this result agrees with observation, and with the elementary theory as well, if it be assumed that the velocity of the aether at the observing station P is equal to the translational velocity of the earth.

2. Velocity Potential in an Incompressible Aether

Here, however, a serious doubt arises, whether the existence of a velocity potential is compatible with the requirement that the velocity should be equal all over the surface of the earth (viz. equal to the earth's velocity in its annual motion), whereas the assumption of such a potential is indispensable for obtaining the correct value of the aberration. As a matter of fact, there is nothing uncommon about a motion for which a velocity potential exists. In a frictionless liquid in which there are, at a given instant, and therefore also at any later time, no vortices, no other motion, in fact, is possible. But, as was just mentioned, the assumption of a velocity potential for an incompressible aether cannot be reconciled with the other requirement that the velocity at the earth's surface should be everywhere the same, in size as well as in direction. In fact, in an incompressible aether the motion is completely determined, if it be assumed that a velocity potential exists and that the normal velocity component of the aether at the surface of the earth is everywhere equal to that of the translation velocity of the earth ; but the tangential components are then found to be different. If the origin is at the centre of the earth, the X-axis in the direction of the translational motion of the earth, and if the velocity of this motion is p, then all requirements are satisfied (and according to the theory of the Laplace equation this is the only possible solution) by the potential

$$\phi = \frac{Cx}{r^3},$$

so that

$$u = C\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right),\ v = -3C\frac{xy}{r^5},\ w = -3C\frac{xz}{r^5},$$

where C is determined by the condition that the normal component of the velocity at the surface of the earth should be equal to the component of p in the same direction. If R be the

radius of the earth, and θ the angle between the direction r and the X-axis, then, since $\partial\phi/\partial r = -2Cx/r^4$, the last condition gives

$$-2C\cos\theta/R^3 = p\cos\theta,$$

and therefore

$$C = -\tfrac{1}{2}pR^3.$$

Now, in order to see that the velocity of the aether cannot be the same all over the surface, viz. equal to that of the earth, it is enough to notice that for points of the surface in a plane through the centre and perpendicular to OX the velocity along the X-axis becomes $-\frac{1}{2}p$, whereas at the intersection point of the surface with the X-axis we find the velocity p. In order to avoid this difficulty, one might perhaps take advantage of the circumstance that the velocity potential need not exist in the whole space around the earth, since we are concerned only with a limited region. This, however, would lead to very artificial and improbable concepts.

3. Planck's Theory. Compressible Aether

It was shown by Planck how these difficulties can be met by giving up the incompressibility of the aether and by assuming that the aether is subject to the earth's attractive force. To investigate the details of the motion, it will be convenient to attribute to the whole universe a velocity $-p$, while considering the earth to be at rest. We have then to investigate the disturbance produced by the fixed sphere in the otherwise uniform aether stream. Planck adjusts his hypotheses so that the velocity at the surface of the sphere should be small. In order to see how this requirement can be satisfied, it is enough to realise that a plane through the centre of the sphere perpendicular to the direction of the stream must be traversed in a given time by the same amount of aether as in the absence of the sphere. The quantity of aether which in the latter case has to stream through the circle cut out by the sphere from that plane, must find its way through the remaining part of the plane surrounding the circle, and especially through that nearest to the circle. This is, with a small velocity, only possible if the density in the neighbourhood of the sphere is large. It is necessary, therefore, to assume that the aether [to be thus condensed] is attracted by the earth. It must further be assumed

that the velocity of light in this very condensed aether is the same as in the aether of normal density. Planck assumes also that the aether while being condensed behaves like a gas.

We will now work out in detail these ideas. Let the Z-axis be chosen in the direction of motion of the aether at infinity. Let, as before, ϕ be the velocity potential and, therefore, the velocity components $u=\partial\phi/\partial x$, $v=\partial\phi/\partial y$, $w=\partial\phi/\partial z$. The distance of a point (x, y, z) from the centre of the earth will be denoted by r, and the radius of the earth by r_0. Let p be the pressure, and k the density of the aether, and let $k/p=\mu$ be assumed to be constant. Further, let V be the potential of the attractive force, per unit mass, so that the components of this force, again per unit mass, are $-\partial V/\partial x$, $-\partial V/\partial y$, $-\partial V/\partial z$. Then $V=q/r$, and the force, which is radial, is, per unit mass, $-\partial V/\partial r=q/r^2$, and therefore at the surface, $q/{r_0}^2=-g$. Thus q is determined, and we have

$$V=-\frac{g{r_0}^2}{r}.$$

Let us now write down the equations of motion for a stationary state, in which the velocity and the density at every point have always the same values. First of all, the equation of continuity,

$$\frac{\partial}{\partial x}\left(k\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial\phi}{\partial z}\right)=0. \quad . \quad . \quad . \quad (1)$$

Next, the equation of motion,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial V}{\partial x}-\frac{1}{k}\frac{\partial p}{\partial x},$$

where the left-hand member represents the acceleration along the X-axis, since, the state being stationary, $\partial u/\partial t=0$. The right-hand member is the force per unit of mass in the same direction. Two similar equations will hold for the Y- and the Z-directions. Owing to the existence of a velocity potential an integral can be found at once. In fact, the equations of motion can be written

$$\tfrac{1}{2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}+\frac{\partial V}{\partial x}+\frac{1}{k}\frac{\partial p}{\partial x}=0,$$

etc., whence

$$\int\frac{dp}{k}+V+\tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}=\text{constant}. \quad . \quad (2)$$

Let us now assume that the term containing the square of the velocity can be neglected. Then,

$$\int \frac{dp}{k} + V = \text{constant.}$$

This equation shows that, to the assumed degree of approximation, the density distribution in the aether can be taken to be the same as in the state of rest. Thus we can write

$$\frac{1}{\mu} \log p - g\frac{r_0^2}{r} = \text{const.},$$

or also

$$\frac{1}{\mu} \log k - g\frac{r_0^2}{r} = \text{const.}$$

The integration constant can be expressed in terms of the aether density k_0 at the surface of the earth,

$$\frac{1}{\mu} \log k - g\frac{r_0^2}{r} = \frac{1}{\mu} \log k_0 - gr_0,$$

instead of which we can write

$$\log \frac{k}{k_0} = \alpha\left(\frac{1}{r} - \frac{1}{r_0}\right),$$

where $$\alpha = \mu g r_0^2.$$

For $r = \infty$ we find the limit value of k,

$$k_\infty = k_0 e^{-\alpha/r_0}.$$

In the next place, Planck determines the velocity potential from the equation (1), where now k is a known function of the co-ordinates. What is required is a solution representing a motion which at large distances reduces to a stream along the Z-axis. Let γ be the velocity of this stream.* Then, at large distances, $\phi = \gamma z$. This suggests the form $\phi = zf(r)$. Then,

$$\frac{\partial \phi}{\partial x} = \frac{xz}{r}\frac{df}{dr}, \quad \frac{\partial \phi}{\partial y} = \frac{yz}{r}\frac{\partial f}{\partial r}, \quad \frac{\partial \phi}{\partial z} = \frac{z^2}{r}\frac{df}{dr} + f,$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{z}{r}\frac{df}{dr} - \frac{x^2 z}{r^3}\frac{df}{dr} + \frac{x^2 z}{r^2}\frac{d^2 f}{dr^2},$$

* This was previously denoted by $-p$. In accordance with Planck's notation p will now be used for the pressure.

$$\frac{\partial^2\phi}{\partial y^2}=\frac{z}{r}\frac{df}{dr}-\frac{y^2z}{r^3}\frac{df}{dr}+\frac{y^2z}{r^2}\frac{d^2f}{dr^2},$$

$$\frac{\partial^2\phi}{\partial z^2}=\frac{3z}{r}\frac{df}{dr}-\frac{z^3}{r^3}\frac{df}{dr}+\frac{z^3}{r^2}\frac{d^2f}{dr^2},$$

and the equation (1) becomes

$$k\left(4\frac{z}{r}\frac{df}{dr}+z\frac{d^2f}{dr^2}\right)+\frac{\partial k}{\partial x}\cdot\frac{xz}{r}\frac{df}{dr}+\frac{\partial k}{\partial y}\cdot\frac{yz}{r}\frac{df}{dr}+\frac{\partial k}{\partial z}\left(\frac{z^2}{r}\frac{df}{dr}+f\right)=0.$$

Substituting here the value of $\log k$ just found, we have

$$4\frac{z}{r}\frac{df}{dr}+z\frac{d^2f}{dr^2}-\alpha\left\{\frac{x}{r^3}\frac{xz}{r}\frac{df}{dr}+\frac{y}{r^3}\frac{yz}{r}\frac{df}{dr}+\frac{z}{r^3}\left(\frac{z^2}{r}\frac{df}{dr}+f\right)\right\}=0,$$

or, after some reductions,

$$\frac{d^2f}{dr^2}+\left(\frac{4}{r}-\frac{\alpha}{r^2}\right)\frac{df}{dr}-\frac{\alpha}{r^3}f=0.$$

The solution of this equation is

$$f=a\left(\frac{\alpha}{2r}-1\right)+b\left(\frac{\alpha}{2r}+1\right)e^{-\alpha/r}. \quad . \quad . \quad . \quad (3)$$

The constants a and b will be determined by considering the state of things at infinity and at the earth's surface. For $r=\infty$ we must have $\phi=z\gamma$, and since $\phi=zf(r)$,

$$f(\infty)=\gamma,$$

and $b-a=\gamma$. At the surface of the earth the aether cannot have any velocity perpendicular to that surface, *i.e.* $\partial\phi/\partial r=0$, and since

$$\frac{\partial\phi}{\partial r}=\frac{z}{r}f(r)+zf'(r),$$

we have, at the earth's surface, $f(r)+rf'(r)=0$. Substituting the value of $f(r)$ and putting $r=r_0$, we find

$$-a+be^{-\alpha/r_0}+b\frac{\alpha}{r_0}\left(\frac{\alpha}{2r_0}+1\right)e^{-\alpha/r_0}=0.$$

Thus

$$a=b\left(\frac{\alpha^2}{2r_0^2}+\frac{\alpha}{r_0}+1\right)e^{-\alpha/r_0}. \quad . \quad . \quad . \quad . \quad (4)$$

Now for the sliding of the aether along the earth's surface. In order to find this, we have to calculate the Z-component of the velocity. From $\phi = zf(r)$ follows

$$\frac{\partial \phi}{\partial z} = f(r) + \frac{z^2}{r} f'(r),$$

and, at the surface of the earth, where $f(r) + rf'(r) = 0$,

$$\frac{\partial \phi}{\partial z} = f(r)\left(1 - \frac{z^2}{r^2}\right) = \sin^2 \theta f(r),$$

where θ is again the angle between r and the Z-axis. If ω is the velocity in the tangential direction, then, since $\partial\phi/\partial z = \omega \sin \theta$, we have

$$\omega = \sin \theta \, . f(r),$$

whence, by (3) and (4),

$$\omega = \sin \theta \left\{ \left(\frac{a^2}{2r_0^2} + \frac{a}{r_0} + 1 \right)\left(\frac{a}{2r_0} - 1 \right) + \frac{a}{2r_0} + 1 \right\} be^{-a/r_0} = \frac{a^3}{4r_0^3} be^{-a/r_0} \sin \theta.$$

Our aim now is to make the coefficient of $\sin \theta$ small in comparison with γ. This will be the case if a/r_0 is large. For then, in virtue of (4), the constant a will be small as compared with b, and therefore, since $b - a = \gamma$, b will be approximately equal to γ.

Planck has made some numerical estimates under different assumptions. Let us first take $\mu = k/p$ and $\alpha = \mu g r_0^2$ as great as for air at 0° C. and for g the value which holds for ordinary ponderable matter. Then the consequences hold for the air actually surrounding the earth. Thus, $a/r_0 = 800$, and the ratio of the density at the earth's surface to that prevailing at a large distance is

$$e^{a/r_0} = e^{800}.$$

The result of the computation is therefore, practically, that our atmosphere can well be kept by the earth, though at the same time the theory shows that some residual sliding is unavoidable, provided that friction be disregarded. Now, with reference to the aether we need not choose our assumptions so as to be driven to such an extravagant condensation. It can easily be calculated how far we must go with this. The aberration constant is known only up to $\frac{1}{2}$ per cent. In other words, the effect of a [sliding] velocity smaller than $\frac{1}{200}$

of the velocity of the earth cannot be detected. It is enough, therefore, that the velocity of the aether at the earth's surface should remain below this limit. For this purpose it is only necessary to have $\alpha/r_0 > 11$. Even so we are still left with a condensation e^{11} [or about 60,000]. The natural question arises whether such a strong condensation could not be tested in an independent way. Notice that the condensation around the sun and similar bodies will be considerably greater, since the attraction is proportional to the mass. The aether condensation due to the presence of the sun will also increase somewhat the condensation e^{α/r_0} at the earth's surface. A number of questions suggest themselves here, but must be left unanswered.

4. Fresnel's Theory. Fixed Aether

We now turn to the alternative hypothesis, that of a fixed aether. This was already assumed by Fresnel. The aether must then be able to pass freely through the earth. The atoms themselves may well be impenetrable, provided, however, that they are assumed to occupy but a small fraction of the total volume. We may also assume that the atoms are not impenetrable, following a line of thought according to which the atoms or their constituents are but special modifications [singularities] of the aether.

In order to explain aberration we again apply Huygens' principle. Let us take the simple case in which the position of a star is being determined by means of a primitive sighting apparatus (without lenses), and let the presence of the atmosphere be disregarded. The explanation is then the same as on the emission theory. The propagation of the wave-front which at a certain instant reaches the aperture AB (Fig. 2) is unaffected by the motion of AB. A simple construction gives us that portion of the screen CD which receives the aether disturbance, and it is manifest that in this way one falls back to the older explanation. But this reasoning cannot be applied when we are concerned with refracting surfaces. In this direction many experiments on aberration were made. As a typical case

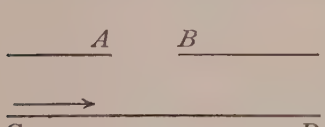

Fig. 2.

we can consider the famous experiment of Arago. If *ab* (Fig. 3) is the direction in which, with the aberration, a star would be seen directly, then *cd*, the direction in which the star is seen through a prism, will coincide with that of the refracted ray belonging to *ba* as incident ray. Another experiment [Boscovich - Airy] proves that, in observing a star, a telescope filled with water has to be set in the same direction as an ordinary telescope. To sum up, these experiments show that all refraction phenomena are the same as if there were no aberration.

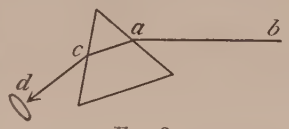

Fig. 3.

5. Dragging Coefficient

In order to explain Arago's experiment Fresnel introduced the dragging-hypothesis, which amounts to this :

Let W be the propagation velocity of light in a given medium, when this is at rest. Then, if the medium is moving with a velocity p in the direction of light propagation, the velocity of light relatively to the aether is, according to Fresnel, not $W+p$ but $W+kp$, where k is a fraction. In other words, the ponderable medium behaves as if it dragged the light with a velocity which is only a certain fraction of its own velocity. The coefficient k must then be

$$k = 1 - \frac{1}{n^2},$$

where n is the refractive index of the medium when at rest. For $n=1$ we have $k=0$, as it should be ; for when n tends to unity, the medium becomes indiscernible from the aether.

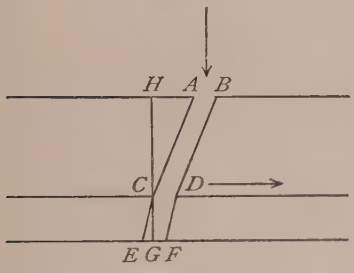

Fig. 4.

This expression for the coefficient k can be easily found by the following reasoning. Let us suppose that the position of a celestial object is being determined by means of a sighting apparatus consisting of a screen with an aperture *AB* (Fig. 4), which is fixed in the aether, and of a second screen with an aperture *CD*, behind which is placed some ponderable medium, as

c.g. glass, and finally of a third screen which receives the light. Since the refraction is such as if the apparent direction of the ray were the true one, the segment EF of the third screen which receives the light will be found by putting

$$\sin(ECG) = \frac{1}{n}\sin(ACH).$$

This is the result of the experiment. Now, let v_0 be the velocity of light in the aether, v_1 the light velocity in glass (at rest), and p the velocity of the earth, and let us consider the case in which the incident light is perpendicular to the first screen. Then, since the deviation from the normal remains small,

$$\sin(ACH) \doteqdot \tan(ACH) = p/v_0,$$

and therefore, $\angle ECG = p/nv_0$, and $EG = lp/nv_0$, if $l = CG$. The segment thus determined differs from that which we would find if, starting with the wave-front CD, we applied Huygens' principle and used in it v_1 as light velocity; for then we should find a segment $E'F'$ such that

$$E'G : CG = p : v_1 = p : v_0/n,$$

and therefore, $E'G = lnp/v_0$. To reconcile this with the experimental result we have to assume that within the time l/v_1 the glass has dragged the light over a distance

$$E'E = E'G - EG = lnp/v_0 - lp/nv_0,$$

that is to say, with the velocity

$$\frac{lp}{v_0}\left(n - \frac{1}{n}\right) : \frac{l}{v_1} = p\left(1 - \frac{1}{n^2}\right).$$

6. Theory of Aberration

If this dragging coefficient is assumed, it can be proved that all the phenomena of refraction, etc., are such as if there were no aberration. For this purpose we use the artifice of imparting to the whole system a velocity equal and opposite to that of the earth. The earth is then at rest, while the aether has everywhere the same velocity. This proof can be given for the more general case of any motion of the aether, provided it has a velocity potential. Our first business is to find out how the wave-fronts and the light-rays are to be determined. The elementary wave-

front spreading out from a point O would, after a time t, in a stationary aether, be a sphere of radius $v_0 t$. Now, if p be the velocity of the aether at O, and if it varies continuously from point to point, then, neglecting infinitesimal terms of the second order, it can be shown that this sphere is simply displaced as a whole over the distance pt. The successive wave-fronts are the envelopes of the spheres thus determined, every time to be constructed around the points of the preceding wave-front as centres, and taking account of the aether velocity at those points.

To find the wave-fronts in a ponderable medium, let us first consider the case in which the aether is at rest, while the ponderable medium has the velocity $-p$. The elementary wave-front is then, after a time t, a sphere of radius $v_1 t$ which is displaced as a whole over a distance $-pt(1-1/n^2)$. If we now give to the whole system the velocity p, the elementary wave-front will be displaced relatively to the ponderable medium, thus brought to rest, over the distance

$$\frac{1}{n^2} pt = \kappa pt,$$

where $\kappa = 1/n^2$. This coefficient can be said to determine to what extent the light has been dragged by the aether. Let now O (Fig. 5) be a point of a wave-front. The elementary wave-front emanating from O is a sphere with O' as centre, where $OO' = \kappa pt$, and of radius $v_1 t$. Let A be the point of contact of this sphere with the envelope which is the new wave-front. Then OA will be an element of the light-ray through O, and we find for the propagation velocity along this ray, relatively to the earth, $w = OA/t$. Thus, if θ be the angle $O'OA$, we have

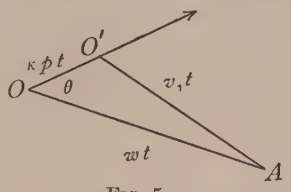

Fig. 5.

$$v_1^2 = w^2 - 2\kappa wp \cos\theta + \kappa^2 p^2,$$

whence, neglecting higher powers of p,

$$w = v_1 + \kappa p \cos\theta + \frac{\kappa^2 p^2}{2v_1}(\cos^2\theta - 1) \quad . \quad . \quad . \quad (5)$$

and

$$\frac{1}{w} = \frac{1}{v_1}\left\{1 - \frac{\kappa p}{v_1}\cos\theta + \frac{\kappa^2 p^2}{2v_1^2}(\cos^2\theta + 1)\right\}. \quad . \quad . \quad (6)$$

These formulae hold for any continuous space-distribution of the velocity p.

Having thus found the velocity along the ray, we can readily determine the light path relatively to the earth. For this purpose we make use of a theorem according to which light follows that path to which corresponds the shortest time, a theorem which in the present case can also be easily deduced from Huygens' principle. In fact, let AB be the light-ray determined by a Huygens construction, and let us consider any other path between A and B. Both paths cut the successive wave-fronts. The intersection points of the latter with the actual light-ray may be called corresponding points. Let S and S' be two successive positions of a wave-front separated by a very short time-interval, and let P', Q', etc., be points of S' corresponding to P, Q, etc., chosen arbitrarily on S. Then the time of passage along PP', QQ', etc., always with the ray velocity belonging to these lines, will be the same. The time required for covering any other path drawn from S to S', again with the ray velocity belonging to it, that is to say, a path laid through non-corresponding points, will be longer. This appears from Fig. 6, in which A is a point of the first wave-front and $A'D$ lies in the second wave-front, A' being the point in which the latter wave-front touches the elementary wave $A'C$ circumscribed around A, so that A' and A are corresponding points. Since the light time for AA' is as long as that for AC, that for AD is evidently longer. Let us now imagine between the points A and B the whole series of successive positions of the wave-front. Then, remembering that the light-ray is the locus of corresponding points, while this cannot be said of any other path from A to B, it will become manifest that the time along the light-ray is indeed the shortest among all paths leading from A to B.

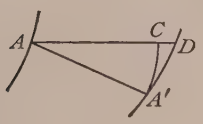

FIG. 6.

This being proved, the light path can be determined in the following way. The time required for covering an arbitrary path between A and B is $\int_A^B ds/w$, the integral to be extended over the path. Now, this expression has to be a minimum for the actual light path.

Let us first take the case of a non-homogeneous (but isotropic) ponderable medium, without surfaces of discontinuity, however.

Then κ, upon which w depends, is variable, since n as well as v_1 are variable. Further, w depends also on p and θ, which in their turn depend on the state of motion of the aether, of which we have only assumed that it has a velocity-potential ϕ.

Thus, taking account of (5), and neglecting terms of the order p^2/v^2, we find for our integral

$$\int_A^B \frac{ds}{v_1}\left(1-\frac{\kappa p}{v_1}\cos\theta\right)=\int_A^B\frac{ds}{v_1}-\int_A^B\frac{1}{n^2}\frac{p_s}{v_1^2}ds$$
$$=\int_A^B\frac{ds}{v_1}-\int_A^B\frac{1}{n^2v_1^2}\frac{\partial\phi}{\partial s}ds.$$

Now, $nv_1=v_0$ being constant throughout the space, the last integral is $\frac{1}{v_0^2}(\phi_B-\phi_A)$, and therefore independent of the path between A and B. The minimum property is thus influenced only by the first integral on the right hand, and since this contains no trace of motion, the light path will be the same as if there were no motion of the aether. Notice that this proof is based, first, on the assumption of Fresnel's dragging coefficient and, second, upon the existence of a velocity-potential.

It follows from the Huygens construction that the minimum property holds also for the case in which the light passes from one to another medium. Consequently, the light path can in this case be found by the same reasoning as before. In fact, considering any path from A to B which cuts the refracting surface at C, and putting $1-n^2v_1^2=\mu$, we have

$$\int_A^B\frac{ds}{w}=\int_A^C\frac{ds}{w}+\int_C^B\frac{ds}{w}$$
$$=\int_A^C\frac{ds}{v_1}-\mu(\phi_C-\phi_A)+\int_C^B\frac{ds}{v_1}-\mu(\phi_B-\phi_C)$$
$$=\int_A^B\frac{ds}{v_1}-\mu(\phi_B-\phi_A).$$

In order that this should be a minimum, it is enough to make the first term a minimum. Consequently, the light-rays in reference to the earth obey the ordinary laws of refraction.

We have still to consider the relation between the light-ray

and the wave-front. For these are no longer perpendicular to each other. Let O (Fig. 7) be a point of a wave-front, let $OO' = p\kappa t$, and let P be the corresponding point of a successive wave-front, so that $OP = wt$ will be an element of the light-ray, and $O'P = v_1 t$. If now the wave-front, and therefore the wave-normal ($O'P$), at every point be given, then the direction of the light-ray can be found by combining the velocity v_1 along the normal with the velocity κp. *Vice versa*, if the ray be given, the direction of the wave-normal will be found by combining the velocity w along the ray with the velocity $-\kappa p$. In the latter construction, and up to terms of the order p^2/v^2, the velocity w can be replaced by v_1 taken in the same direction.

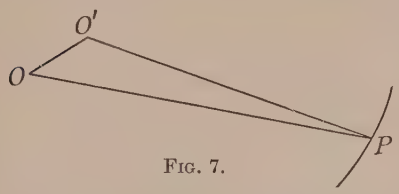

FIG. 7.

Let us still consider a luminous point L in a homogeneous medium, and let us determine the wave-fronts by the indicated construction. The light-rays being straight lines diverging from L, it is required to find the direction of the wave-normal at any point A. If x, y, z be the co-ordinates of A, with L as origin, the components of the ray velocity v_1 at the point A are

$$\frac{x}{r}v_1,\ \frac{y}{r}v_1,\ \frac{z}{r}v_1,$$

and these must be compounded with $-\kappa p$, the velocity of dragging. The components of the latter are $-\kappa\partial\phi/\partial x$, $-\kappa\partial\phi/\partial y$, $-\kappa\partial\phi/\partial z$. Thus the components of the resultant will be

$$\frac{\partial}{\partial x}(v_1 r - \kappa\phi),\ \frac{\partial}{\partial y}(v_1 r - \kappa\phi),\ \frac{\partial}{\partial z}(v_1 r - \kappa\phi),$$

and since this resultant has the direction of the wave-normal, the equation of the wave-front becomes

$$v_1 r - \kappa\phi = \text{const.}$$

In the neighbourhood of L the potential ϕ can be considered as a linear function of the co-ordinates, so that the last equation assumes the form

$$v_1 r - \kappa(ax + by + cz) = \text{const.}$$

As we already know, the wave-fronts in the immediate neighbourhood of L are spheres whose centres lie at a certain distance from L. It is manifest that the equation agrees with this, provided terms with p^2 are neglected, as in fact they were in deducing it. Again, waves converging towards a point have the same form as would have expanding ones, if the velocities in the aether were everywhere reversed.

In order to apply this reasoning to the determination of the direction in which we see a star through a telescope, let us consider a wave-front arriving from a star, and let us derive from the given wave-normal the direction of the light-ray. It is these relative light-rays which we observe in our experiments. In the free aether, far away from the earth, the direction of the relative ray is found by combining the velocity v_0 along the wave-normal with κp, where $\kappa = 1$. In this manner we find, obviously, the same direction as according to the elementary theory. The further progress of these relative rays can now be followed up by the ordinary laws of refraction, etc. And we have now to orient the telescope so that the rays thus treated should converge upon the intersection point of the cross-wires. In this way the theory accounts for all the experimental facts.

Also the diffraction and interference phenomena are such as if the earth were at rest and as if we had to do with the velocity v_1. If we have, for instance, two paths between A and B, the interference will depend only on the difference of the light times along them, and this difference can be determined by using for each path element the velocity v_1. For this amounts only to omitting, for the two paths, the term $\mu(\phi_A - \phi_B)$, which is the same for both. Here again it is only necessary to assume that there exists a velocity-potential, but not that the earth is penetrable for the aether. The latter, however, must be assumed for all transparent media in order to have a reasonable explanation of the dragging coefficient. The simplest, after all, is Fresnel's theory.

7. Michelson's Experiment

We have thus far neglected all terms having p^2/v^2 as factor. See formula (5).

In certain interference experiments, however, the accuracy can be pushed so far that these second-order terms have to be

taken into account. Such is the famous experiment of Michelson, already suggested by Maxwell. Let A and B be two points fixed on the earth, and let the latter move with the velocity p along AB. What is the time taken by light for a complete to-and-fro passage? This will depend on whether the aether shares in the earth's motion or not. In the former case the time in question will be independent of the velocity p, but in the latter case the velocity of light relative to the earth will be $v-p$ in one, and $v+p$ in the opposite direction, and therefore the time required for a complete to-and-fro passage [up to fourth-order terms],

$$\frac{l}{v-p}+\frac{l}{v+p}=\frac{2l}{v}+2l\frac{p^2}{v^3}.$$

Since Michelson determines this time by means of an interference phenomenon, the influence of the last term will still be perceptible, though it amounts only to a fraction of the vibration period. Let us see how large l must be in order to give an observable effect. The shift of the interference fringes due to a time lag of $\frac{1}{10}$ of the vibration period T will be just detectible. For this purpose we must have

$$2l\frac{p^2}{v^3}=\frac{1}{10}T,$$

hence, if λ be the wave-length of the light,

$$2l=\frac{\lambda}{10}\frac{v^2}{p^2}=10^7\lambda=600 \text{ cm.},$$

or $l=3$ metres.

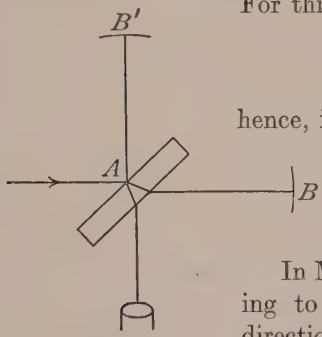

FIG. 8.

In Michelson's experiment a ray traversing to and fro a certain distance in the direction of the earth's motion is made to interfere with a ray traversing the same distance up and down in a perpendicular direction. Fig. 8 gives a schematic representation of the apparatus. This consists of two fixed mirrors B and B' and a dividing glass plate A, inclined at 45° to the incident light beam. A part of this beam passes through the plate, is reflected at B, and after a reflection at the lower face of the plate enters the telescope. Another part of the incident light is reflected at A towards the mirror B' and thence through the glass plate into the telescope, where it interferes with the first partial beam. The experiment consists

in turning the whole apparatus, including the light source and the telescope, by 90° and comparing the interference fringes in the new and the original orientation. The effect of the phase difference is thus doubled. Yet no displacement of the interference fringes was observed.

Let us consider the theory of this experiment somewhat more thoroughly. We have to compare the times taken by light to traverse two different paths between two points A and B (Fig. 9). We have already seen that, if only first-order terms are retained, the difference of these times is the same as if the earth were at rest. Now, to determine the effect of second-order terms we must take into account that the light path between A and B is itself slightly changed by the motion. In fact, the light path could be identified with that corresponding to a fixed earth only when we confined ourselves to terms of the order p/v. Now, however, the slight change of the path implying second-order terms need not be negligible. But it can be easily shown that even with the inclusion of second-order effects the propagation time from A to B can be calculated with sufficient accuracy by extending the integral $\int ds/w$, with w given by (6), along the light path which would correspond to a fixed earth. Let, *e.g.*, ACB (Fig. 9) be the light path with the earth at rest, and ADB the actual light path for a moving earth. Then the integral $\int ds/w$ would have to be evaluated for the path ADB. But since the value of this integral is just a minimum, it will differ from that taken along the path ACB only by a quantity of the second order in the deviations of the two paths, such as CD. And since, as already mentioned, this deviation is itself of the order p^2/v^2, the difference between the integrals extended over ACB and ADB will be of the order p^4/v^4, and can thus be neglected.

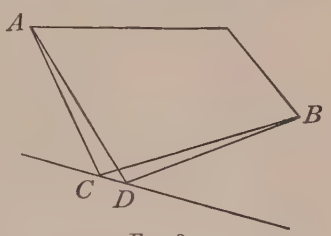

Fig. 9.

The same holds also for the other light pencil, as indicated in the figure by the lines without letters.

The propagation time from A to B, $\tau = \int ds/w$, can now be calculated as in Art. 6, developing $1/w$ into a power series of p/v and confining ourselves to the first three terms.

Thus, $\tau = \tau_1 + \tau_2 + \tau_3$, where, by (6),

$$\tau_1 = \int \frac{ds}{v_1},$$

$$\tau_2 = -\int \kappa \frac{p \cos\theta}{v_1^2} ds$$

(which has no influence on the phase difference), and

$$\tau_3 = \tfrac{1}{2}\int \frac{\kappa^2 p^2}{v_1^3}(1+\cos^2\theta) ds.$$

In Michelson's experiment we can put $\kappa = 1$, and p is, by Fresnel's theory, the velocity of the earth. To the phase change due to the turning of the apparatus the integral τ_2 contributes nothing, so that only τ_3 has to be taken care of, where $\theta = 0°$ or $180°$ for the first, and $\theta = 90°$ or $270°$ for the second of the interfering rays. If l be the doubly covered path, then, as we already saw,

$$\tau_3 = 2p^2 l/v^3$$

for the first ray, and $\tau_3 = p^2 l/v^3$ for the second ray. Thus also the latter is somewhat affected by the earth's motion. This can also be seen in the following simple way. A ray impinging upon the glass plate at A (Fig. 10) returns from the mirror B and meets it again in A', so that the distance over which the plate was displaced in the meantime is $AA' = 2lp/v$. Consequently the length of the light path is $2\sqrt{l^2 + l^2p^2/v^2} = 2l(1 + p^2/2v^2)$. The difference, as compared with a fixed earth, is therefore lp^2/v^2, and the difference of the corresponding time, lp^2/v^3, which is the value just found for τ_3.

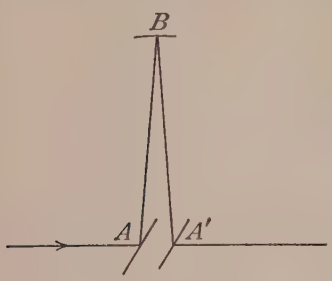

FIG. 10.

As Michelson overlooked the influence of the earth's motion upon the light time along the path ABA', in his first experiment he estimated the theoretical effect twice too high. In this case the distance l was 1·2 metre, so that a shift of $\frac{1}{10}$ of a fringe width was expected. The corrected theory gave only $\frac{1}{20}$ of a fringe width, and this was below the threshold of reliable observability. Michelson, therefore, repeated the experiment

with the modification that each of the two beams passed several times between the mirrors. In this manner the light path was increased to 22 metres and should have given a displacement of $\frac{4}{100}$ of a fringe width (an estimate in which account was also taken of the motion of the solar system). But even now the result of the experiment was negative in contradiction to Fresnel's theory. It will be kept in mind that the validity of this theory is not limited to the case of an aether at rest as a whole, but extends also to types of motion of the aether for which there exists a velocity-potential.

This experiment would also clash with any theory which attributes to the relative velocity of the aether and the earth a value not sensibly smaller than the translation velocity of the earth. In what follows, however, we shall have in mind only Fresnel's theory.

8. Contraction in the Direction of Motion

We can explain the negative result of Michelson's experiment by assuming that the length of the arms of the apparatus is changed by turning it through a right angle. This change can be assumed to be just such as to give to τ_3 the same value for both rays. This calls for a contraction of the path in the direction of motion, as compared with the perpendicular path, such that the corresponding light time should be shortened by p^2l/v^3. The path then has to be shortened by lp^2/v^2, and therefore l by $\frac{1}{2}\frac{p^2}{v^2}l$. This dependence of the dimensions upon the orientation with respect to the earth's motion is not as strange as it might seem at first. In fact, the dimensions are determined by molecular forces, and since these are transmitted through the aether, it would rather be surprising if its state of motion had no influence upon the dimensions of bodies. The nature of the molecular forces is not known to us. Yet, if we suppose that they are transmitted through the aether in the same way as electric forces, we can develop the theory of this contraction, and we then find for its amount just what is required for the explanation of the nil-effect of Michelson's experiment. This contraction would amount for the diameter of the earth to 6·5 cm., and for a metre rod to $\frac{1}{200}$ of a micron.

II

MECHANICAL AETHER THEORIES

9. Maxwell's Equations

We will now consider some theories of the nature of the aether. Such a theory must, in the first place, explain the electromagnetic phenomena. We shall, therefore, begin with an interpretation of Maxwell's equations. Here one can put different requirements. One can content himself with a theory accounting for the phenomena in isotropic and homogeneous media, or try to include also the anisotropic and heterogeneous media, and so on. To begin with, we shall exclude the conductors only, and shall thus consider anisotropic as well as non-homogeneous dielectrics. The latter will enable us to treat the boundary conditions, and therefore also such phenomena as reflection and refraction.

The magnitudes appearing in Maxwell's equations are: the electric force **E**, the dielectric displacement **D**, the magnetic force **H**, and the magnetic induction **B**. The equations are

$$\text{rot } \mathbf{H} = \frac{1}{c}\dot{\mathbf{D}}, \text{ or}$$

$$\left.\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{1}{c}\frac{\partial D_x}{\partial t} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{1}{c}\frac{\partial D_y}{\partial t} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{1}{c}\frac{\partial D_z}{\partial t} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{div } \mathbf{D} = 0, \text{ or } \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0, \quad . \quad . \quad . \quad (8)$$

and

$$\text{rot } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \text{ or}$$

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=-\frac{1}{c}\frac{\partial B_x}{\partial t}\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=-\frac{1}{c}\frac{\partial B_y}{\partial t}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=-\frac{1}{c}\frac{\partial B_z}{\partial t}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{div } \mathbf{B}=0, \text{ or } \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0. \quad . \quad . \quad . \quad (10)$$

Further, we assume linear relations between the components of **D** and **E** and between those of **B** and **H**,

$$\begin{aligned}D_x&=\epsilon_{11}E_x+\epsilon_{12}E_y+\epsilon_{13}E_z, & B_x&=\mu_{11}H_x+\mu_{12}H_y+\mu_{13}H_z,\\ D_y&=\epsilon_{21}E_x+\epsilon_{22}E_y+\epsilon_{23}E_z, & B_y&=\mu_{21}H_x+\mu_{22}H_y+\mu_{23}H_z,\\ D_z&=\epsilon_{31}E_x+\epsilon_{32}E_y+\epsilon_{33}E_z, & B_z&=\mu_{31}H_x+\mu_{32}H_y+\mu_{33}H_z,\end{aligned}$$

where

$$\epsilon_{12}=\epsilon_{21},\ \epsilon_{23}=\epsilon_{32},\ \epsilon_{31}=\epsilon_{13}; \quad \mu_{12}=\mu_{21},\ \mu_{23}=\mu_{32},\ \mu_{31}=\mu_{13}.$$

For isotropic bodies these coefficients are all zero, and

$$\epsilon_{11}=\epsilon_{22}=\epsilon_{33}=\epsilon, \quad \mu_{11}=\mu_{22}=\mu_{33}=\mu.$$

We note that the usual kinematics of continuous media gives equations of the same form as $\text{rot } \mathbf{H}=\frac{1}{c}\dot{\mathbf{D}}$. In fact, if every point of a medium is displaced through an infinitesimal distance, of which the components ξ, η, ζ along three axes can be considered, within a small region, as linear functions of the co-ordinates x, y, z, then the complete change in the neighbourhood of a point can be represented as consisting of a displacement, a rotation, and three dilatations or contractions in three mutually perpendicular directions. To see this, it is enough to remember that a sphere is transformed into an ellipsoid, whose conjugated diameters correspond to orthogonal pairs of diameters of the sphere. This holds also in particular for the principal axes of the ellipsoid. We imagine the sphere to be first displaced so as to bring its centre into coincidence with that of the ellipsoid, then find those diameters of the sphere which coincide with the axes of the ellipsoid, bring them by a rotation into their actual position, and finally give to these diameters by a dilatation or

contraction the actual length of the axes of the ellipsoid. The components of the rotation are

$$p=\tfrac{1}{2}\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right),\ q=\tfrac{1}{2}\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right),\ r=\tfrac{1}{2}\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right). \quad . \quad (11)$$

In a continued displacement we can compare the state at the time t with that at $t+dt$, and the last formulae give us then the connection of the components of the angular with those of the displacement velocity.

We may now try to interpret one group of Maxwell's equations by taking the components of the magnetic force to be proportional to the displacements in the aether, or assuming that the essence of that force consists in such displacements. In symbols, let us put

$$H_x=m\xi,\ H_y=m\eta,\ H_z=m\zeta.$$

Then
$$\frac{\partial D_x}{\partial t}=2cmp,\ \frac{\partial D_y}{\partial t}=2cmq,\ \frac{\partial D_z}{\partial t}=2cmr.$$

Thus, if the observed magnetic force is in its essence a displacement of the aether particles, proportional to it and having the same direction, the nature of the dielectric displacement current consists in rotations which are the result of the aether displacement. But we are then checked by the difficulty that there is in such a picture no place for a constant dielectric displacement, such as occurs always in electrostatics. Moreover, in a constant electric field without magnetic force the displacements ξ, η, ζ vanish, so that there is nothing to distinguish such a state of the medium from one in which an electric field is absent.

10. The Magnetic Force as Velocity. Neumann's Theory of Light

We can avoid this difficulty by identifying with the aether rotations not the displacement current but the dielectric displacement itself. Then our equations have to be made to agree with

$$\frac{\partial p}{\partial t}=\tfrac{1}{2}\left(\frac{\partial^2\zeta}{\partial y\partial t}-\frac{\partial^2\eta}{\partial z\partial t}\right),\ \text{etc.}$$

Consequently, we have to put

$$H_x = m\frac{\partial \xi}{\partial t},\ H_y = m\frac{\partial \eta}{\partial t},\ H_z = m\frac{\partial \zeta}{\partial t}, \quad . \quad . \quad . \quad (12)$$

$$D_x = 2cmp,\ D_y = 2cmq,\ D_z = 2cmr. \quad . \quad . \quad (13)$$

Thus, wherever there is a magnetic force, we must imagine an aether velocity in the direction of this force and proportional to it, and we have to look for the dielectric displacement in the rotation due to or associated with that velocity. In a permanent magnetic field, as *e.g.* around a steel magnet, we have thus to imagine the aether streaming along the lines of force.

We may also notice that in such a case of continual motion the displacements ξ, η, ζ would not remain infinitesimal.

It is important to keep in mind that the interpretation here given implies necessarily that the coefficient m has also in a non-homogeneous medium throughout the same value. Otherwise the values (12) and (13) would not satisfy the equations (7). The expression

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z},$$

for instance, would no longer have the value $2m\partial p/\partial t$, since to this the terms

$$\frac{\partial m}{\partial y}\frac{\partial \zeta}{\partial t} - \frac{\partial m}{\partial z}\frac{\partial \eta}{\partial t}$$

would have to be added. This difficulty would assert itself especially at such places where the properties of the medium, and therefore also the value of m, vary rapidly from point to point, as at the boundary of two media. This theory, therefore, implies that the coefficient of proportionality between the velocity of the aether particles and the magnetic force is always and everywhere the same.

We have still to explain the second group, (9), of equations. Before doing so we have to consider the energy relations. The magnetic energy is a quadratic function of the magnetic force, and has thus to be interpreted as kinetic energy, while the electric energy will become potential energy. The kinetic energy per unit volume, with ρ as density, is

$$\tfrac{1}{2}\rho\left\{\left(\frac{\partial \xi}{\partial t}\right)^2 + \left(\frac{\partial \eta}{\partial t}\right)^2 + \left(\frac{\partial \zeta}{\partial t}\right)^2\right\},$$

and the magnetic energy per unit volume of an isotropic medium, with $\mu = 1$,

$$\tfrac{1}{2}H^2 = \tfrac{1}{2}m^2\left\{\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\zeta}{\partial t}\right)^2\right\}.$$

Thus we have to assume $\rho = m^2$, so that also the density will be everywhere the same. But the case is different if μ is not equal to 1. In such media the magnetic energy per unit volume is

$$\tfrac{1}{2}\mathbf{BH} = \tfrac{1}{2}\mu m^2\left\{\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\zeta}{\partial t}\right)^2\right\},$$

and therefore $\rho = \mu m^2$, *i.e.* the density must be proportional to the permeability. According to this theory the characteristic feature of iron, for instance, is a large aether density within it. For anisotropic media the relations are somewhat more complicated, inasmuch as for these the magnetic energy per unit volume is

$$\tfrac{1}{2}(\mu_{11}H_x^2 + \mu_{22}H_y^2 + \mu_{33}H_z^2 + 2\mu_{23}H_yH_z + 2\mu_{31}H_zH_x + 2\mu_{12}H_xH_y).$$

Now, if we put here $H_x = m\partial\xi/\partial t$, etc., then we do not obtain an expression representing the squared velocity multiplied by a certain factor. By a proper choice of the co-ordinate system the last expression can be transformed into

$$\tfrac{1}{2}(\mu_{11}'H_{x'}^2 + \mu_{22}'H_{y'}^2 + \mu_{33}'H_{z'}^2),$$

and this can be interpreted as kinetic energy per unit volume, if it be assumed that the aether behaves as if it had different mass densities for motions in different directions.

Let us still consider the electric energy which has to be interpreted as potential energy. This is, per unit volume of an isotropic medium,

$$\tfrac{1}{2}\mathbf{DE} = \frac{1}{2\epsilon}D^2 = \frac{1}{2\epsilon}4c^2m^2(p^2 + q^2 + r^2).$$

The aether has thus to be attributed the property that its potential energy is proportional to the square of the rotation of its particles. The question how we have to imagine such an aether will be taken up in the sequel.

This theory resembles the light theory of Neumann, who also assumed that the aether density is the same in all media, and thence deduced that in polarised light the oscillations are in the

plane of polarisation. This means, with the present interpretation, that the magnetic force is contained in the plane of polarisation, as in fact has to be assumed in the electromagnetic theory of light.

11. The Electric Force as Velocity. Fresnel's Theory of Light

As an alternative we might have interpreted kinematically the equations

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c}\frac{\partial B_x}{\partial t}, \text{ etc.}$$

This is entirely analogous to what precedes, the magnetic induction being now represented by the rotation of the aether particles and the electric force by their velocity. The formulae now become

$$\left.\begin{aligned} &E_x = m'\frac{\partial \xi}{\partial t},\ E_y = m'\frac{\partial \eta}{\partial t},\ E_z = m'\frac{\partial \zeta}{\partial t} \\ &B_x = -2cm'p,\ B_y = -2cm'q,\ B_z = -2cm'r \end{aligned}\right\}, \quad . \quad (14)$$

where m' is again a constant. There is something strange in having to consider the electric force as a velocity. For an ordinary conductor carrying a constant charge we would now have a permanent, outward or inward, stream of aether. In the previous theory the continuous streaming of the aether near a magnet was at least circuitous, which state of things it was easy to imagine to last invariably for any length of time.

The electric energy must now be correlated with the kinetic one. Its amount per unit volume, which is $\frac{1}{2}\epsilon E^2$, has to be made equal to

$$\tfrac{1}{2}\rho\left\{\left(\frac{\partial \xi}{\partial t}\right)^2 + \left(\frac{\partial \eta}{\partial t}\right)^2 + \left(\frac{\partial \zeta}{\partial t}\right)^2\right\},$$

so that $\rho = \epsilon m'^2$. Since m' is constant and ϵ has for different dielectrics widely differing values, the density of the aether which is proportional to the specific inductive capacity must have very different values in different substances. As the previous theory approached that of Neumann, the present one resembles Fresnel's theory. In the latter it is assumed that the displacements of the aether particles are perpendicular to the plane of polarisation. This means, with the present interpretation, that such also is the

orientation of the electric force. Thus on either interpretation we remain in harmony with what is generally assumed about the light vibrations in the electromagnetic theory. With regard to this coincidence of the model with Fresnel's theory, we may still notice that to different specific inductive capacities correspond different propagation velocities of the electromagnetic disturbances, while Fresnel looked for the explanation of the different propagation velocities of light in the different densities of the aether within different media, and made this density proportional to the square of the refractive index, which amounts exactly to what we have assumed. With the interpretation treated in this section the electrically anisotropic media offer the same difficulties as did the magnetically anisotropic bodies in the previous one.

For the interpretation of the second set of equations, (7), we consider the magnetic energy which has to be identified with the potential energy of the aether. It is a quadratic function of the rotation components p, q, r. Thus, as in the previous theory, we have to imagine a mechanism whose potential energy is just such a function.

12. Theory of Elasticity. McCullagh's Aether

In an ordinary elastic body the relations are entirely different. Let ξ, η, ζ be again the displacement components (functions of the co-ordinates x, y, z). Then we have, in addition to the displacement and rotation, the dilatations determined by

$$x_x = \frac{\partial \xi}{\partial x},\ y_y = \frac{\partial \eta}{\partial y},\ z_z = \frac{\partial \zeta}{\partial z}, \qquad . \quad . \quad . \quad (15)$$

and the shears expressed by

$$y_z = z_y = \frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y},\ z_x = x_z = \frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z},\ x_y = y_x = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}. \quad . \quad (16)$$

The normal tension components [stresses] are determined by

$$X_x = 2K\{x_x + \theta(x_x + y_y + z_z)\},\ Y_y = 2K\{y_y + \theta(x_x + y_y + z_z)\},$$
$$Z_z = 2K\{z_z + \theta(x_x + y_y + z_z)\},$$

and the tangential components by

$$Y_z = Z_y = Ky_z,\ Z_x = X_z = Kz_x,\ X_y = Y_x = Kx_y, \qquad . \quad (17)$$

where K and θ are the coefficients introduced by Kirchhoff. The energy per unit volume is

$$K(x_x^2 + y_y^2 + z_z^2) + K\theta(x_x + y_y + z_z)^2 + \tfrac{1}{2}K(x_y^2 + y_z^2 + z_x^2). \quad (18)$$

Now, for the aether this must be a quadratic function of the components of rotation,

$$p = \tfrac{1}{2}\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right),\ q = \tfrac{1}{2}\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right),\ r = \tfrac{1}{2}\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right).$$

The next task would be (and such in fact is always the position in the older light theories) to make such assumptions as to give the energy that form.

The most radical means for obtaining this result is to assume for the aether the validity of a very peculiar elasticity theory, such that the potential energy should in every case be a quadratic function of the angles of rotation. This is *McCullagh's aether*. Let us see how this can be done. Elasticity cannot exist if there are no forces tending to bring back the displaced particles to their original position (state of equilibrium). What we are concerned with is the potential energy opposed to these forces. If that energy is to depend only upon the rotations, then the system of those counteracting forces has to consist only of couples. The components of the rotation being p, q, r, we have to assume for the components of the moment of the restoring couple, per unit volume,

$$M_x = -(\alpha_{11}p + \alpha_{12}q + \alpha_{13}r),\ M_y = -(\alpha_{21}p + \alpha_{22}q + \alpha_{23}r),$$
$$M_z = -(\alpha_{31}p + \alpha_{32}q + \alpha_{33}r), \quad . \quad . \quad . \quad (19)$$

where the α's are constants satisfying the conditions $\alpha_{12} = \alpha_{21}$, $\alpha_{23} = \alpha_{32}$, $\alpha_{31} = \alpha_{13}$. These formulae hold also for anisotropic bodies. The peculiar feature of the case in hand is that this couple of forces should not be produced by the neighbouring aether particles. For such a couple would depend only on the relative rotation.

13. Quasi-rigid Aether

To explain these couples we may introduce a second medium which remains in its place, not sharing in the rotation of the first medium, and which exerts upon the latter those restitutive forces.

In the first medium we must then have also some inner forces which do not all vanish. To see this let us recall how in the ordinary cases the relations $Y_z = Z_y$, etc., are deduced. Let us consider a parallelepipedon $dxdydz$ and find the couple which produces a rotation about the X-axis (Fig. 11). The two forces parallel to the Z-axis, acting upon the faces $dxdz$, give the couple $Z_y dxdzdy$, and those parallel to the Y-axis applied to the faces $dxdy$ yield $-Y_z dxdydz$; in all, $(Z_y - Y_z)dxdydz$. Now, this couple together with $M_x dxdydz$ must vanish, otherwise it would give an infinite angular acceleration, since it is of the third order, while the moment of inertia of the parallelepipedon is infinitely small of the fifth order. Therefore, for our aether,

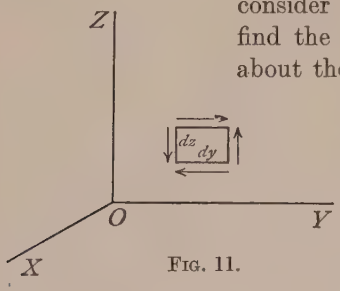

FIG. 11.

$$Z_y - Y_z = -M_x.$$

Thus there must exist tangential stresses. Similarly, of course, we find

$$X_z - Z_x = -M_y,$$
$$Y_x - X_y = -M_z.$$

Moreover, we will make our assumptions as simple as possible. To this end we make the normal stress components zero, $X_x = Y_y = Z_z = 0$, and take for the tangential stresses

$$-Z_y = Y_z = \tfrac{1}{2}M_x,$$
$$-X_z = Z_x = \tfrac{1}{2}M_y,$$
$$-Y_x = X_y = \tfrac{1}{2}M_z.$$

In order to see that these assumptions are in agreement with the adopted energy expression, it is enough to calculate the energy of the medium within a given surface for a stationary state. Let $d\sigma$ be an element of this surface whose outward normal makes with the axes the angles λ, μ, ν. Let us imagine that the displacements ξ, η, ζ are produced by some external forces upon the surface elements, and that these forces as well as the displacements and stresses gradually mount to their final values.

The force acting upon the element $d\sigma$ in the direction of the X-axis is

$$(X_x \cos\lambda + X_y \cos\mu + X_z \cos\nu)d\sigma,$$

and similarly for the Y- and Z-components. Since all the magnitudes increase proportionally to each other, we find for the total work upon the element $d\sigma$

$$\tfrac{1}{2}\{\xi(X_x \cos\lambda + X_y \cos\mu + X_z \cos\nu) + \eta(Y_x \cos\lambda + Y_y \cos\mu + Y_z \cos\nu) + \zeta(Z_x \cos\lambda + Z_y \cos\mu + Z_z \cos\nu)\}d\sigma.$$

A simple reasoning will show that the energy contained within the surface σ is given by this expression integrated over the surface. This integral can be transformed into the volume integral

$$\tfrac{1}{2}\int\left\{\frac{\partial}{\partial x}(X_x\xi + Y_x\eta + Z_x\zeta) + \frac{\partial}{\partial y}(X_y\xi + Y_y\eta + Z_y\zeta) + \frac{\partial}{\partial z}(X_z\xi + Y_z\eta + Z_z\zeta)\right\}d\tau,$$

$d\tau$ being the element of the enclosed volume. Thus, the energy per unit volume is

$$\tfrac{1}{2}\left\{\frac{\partial}{\partial x}(X_x\xi + Y_x\eta + Z_x\zeta) + \frac{\partial}{\partial y}(X_y\xi + Y_y\eta + Z_y\zeta) + \frac{\partial}{\partial z}(X_z\xi + Y_z\eta + Z_z\zeta)\right\}, \quad (20)$$

or, by (11), (15), and (16),

$$\tfrac{1}{2}\{X_x x_x + Y_y y_y + Z_z z_z + \tfrac{1}{2}(Y_x + X_y)x_y + (Y_x - X_y)r + \tfrac{1}{2}(Z_y + Y_z)y_z + (Z_y - Y_z)p + \tfrac{1}{2}(X_z + Z_x)z_x + (X_z - Z_x)q\}. \quad (21)$$

In fact, we have

$$Y_x\frac{\partial\eta}{\partial x} + X_y\frac{\partial\xi}{\partial y} = \tfrac{1}{2}(Y_x + X_y)\left(\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\right) + \tfrac{1}{2}(Y_x - X_y)\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right),$$

etc., and in order to see that the remaining terms disappear, it is enough to remember that we are dealing with a stationary state. Thus, *e.g.*, the coefficient of ξ in (20) becomes

$$\tfrac{1}{2}\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) = \tfrac{1}{2}\rho\frac{\partial^2\xi}{\partial t^2} = 0.$$

Now, if the energy (21) is to depend on the rotations only, then in the first place the terms containing the dilatations must

vanish, *i.e.* $X_x = Y_y = Z_z = 0$, and in order to get rid of the influence of the shears, we must have $Y_x + Y_y = 0$, $Z_y + Y_z = 0$, $X_z + Z_x = 0$. Thus we fall back upon the previous values of the stresses.

The energy per unit volume now becomes

$$-\tfrac{1}{2}(M_x p + M_y q + M_z r)$$
$$= \tfrac{1}{2}(a_{11}p^2 + a_{22}q^2 + a_{33}r^2 + 2a_{12}pq + 2a_{23}qr + 2a_{31}rp),$$

and it remains to be seen how in the two developed theories the coefficients a are correlated with the magnitudes appearing in the electromagnetic equations. In the first case (Art. 10) we had

$$\left.\begin{aligned} H_x = m\frac{\partial \xi}{\partial t},\ H_y = m\frac{\partial \eta}{\partial t},\ H_z = m\frac{\partial \zeta}{\partial t} \\ D_x = 2cmp,\ D_y = 2cmq,\ D_z = 2cmr \end{aligned}\right\}, \quad . \quad . \quad (22)$$

and the aether density was $\rho = m^2$.

The potential energy must coincide with the electric energy. The latter is, per unit volume,

$$\tfrac{1}{2}(\epsilon'_{11}D_x^2 + \text{etc.} + 2\epsilon'_{12}D_xD_y + \text{etc.}),$$

since $$E_x = \epsilon'_{11}D_x + \epsilon'_{12}D_y + \epsilon'_{13}D_z, \text{ etc.}$$

Thus the required coincidence calls for the following relations :

$$a_{11} = 4\epsilon'_{11}c^2m^2,\ a_{22} = 4\epsilon'_{22}c^2m^2,\ a_{33} = 4\epsilon'_{33}c^2m^2,$$
$$a_{12} = 4\epsilon'_{12}c^2m^2,\ a_{23} = 4\epsilon'_{23}c^2m^2,\ a_{31} = 4\epsilon'_{31}c^2m^2.$$

For isotropic media we shall simply have $\mathbf{E} = \epsilon'\mathbf{D}$, $a = 4\epsilon' c^2 m^2$.

The second set of equations (9) now follows from the equations of motion of the aether,

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = \rho\frac{\partial^2 \xi}{\partial t^2}, \text{ etc.}$$

In fact, putting here

$$X_x = 0,\ X_y = \tfrac{1}{2}M_z,\ X_z = -\tfrac{1}{2}M_y,$$

we have $$\tfrac{1}{2}\left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right) = \rho\frac{\partial^2 \xi}{\partial t^2}.$$

Now,

$$M_x = -\frac{4c^2m^2}{2cm}(\epsilon'_{11}D_x + \epsilon'_{12}D_y + \epsilon'_{13}D_z) = -2cmE_x, \text{ etc.}$$

Substituting this and replacing $\partial\xi/\partial t$ by H_x/m, we find

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{1}{c}\frac{\partial H_x}{\partial t},$$

which is the first equation of the second set (for the sub-case $\mu=1$).

Similarly, in the second case, the set (7) of equations can be deduced from the equations of motion of the aether by considering the magnetic energy.

Attention to such an aether was drawn by Kelvin, who called it *the quasi-rigid* aether. With its aid he proposed to account for the magnetic phenomena, and considered, therefore, the magnetic induction as the rotation of the aether elements, which coincides with our second case.

14. Quasi-labile Aether

An explanation of the phenomena can also be arrived at without making about the elasticity of the aether such uncommon assumptions. It will be enough to consider the case of an isotropic homogeneous medium. Let us substitute in the equations of motion

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=\rho\frac{\partial^2\xi}{\partial t^2},\ \text{etc.},$$

the stress components as given by the ordinary theory of elasticity, to wit,

$$X_x=2K\left\{\frac{\partial\xi}{\partial x}+\theta\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)\right\},$$

$$X_y=K\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right),\ X_z=K\left(\frac{\partial\xi}{\partial z}+\frac{\partial\zeta}{\partial x}\right).$$

Then the result will be

$$K\left\{\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)+(1+2\theta)\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)\right\}=\rho\frac{\partial^2\xi}{\partial t^2},\ \text{etc.}$$

We have now to make such assumptions that the left-hand members of these equations should become the rotation

components of a vector. This will be the case if we put $\theta = -1$. Thus, in fact, the last equation will become

$$K\left\{\frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial y}-\frac{\partial \eta}{\partial x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial \zeta}{\partial x}-\frac{\partial \xi}{\partial z}\right)\right\}=\rho\frac{\partial^2 \xi}{\partial t^2},$$

or, in terms of the rotations,

$$K\left(\frac{\partial r}{\partial y}-\frac{\partial q}{\partial z}\right)=-\tfrac{1}{2}\rho\frac{\partial^2 \xi}{\partial t^2},$$

which is the required form.

With the interpretation of Art. 10 this equation is transformed into the first of the equations (9) for an isotropic medium, and with the interpretation of Art. 11 into the first of (7).

The medium just considered is Kelvin's *quasi-labile aether*. The equilibrium of this aether is labile, the potential energy in the state of equilibrium not being a minimum. In fact, it is not difficult to see that the potential energy can under the stated circumstances become negative. For its expression, per unit volume, is

$$K(x_x^2+y_z^2+z_z^2)+K\theta(x_x+y_y+z_z)^2+\tfrac{1}{2}K(x_y^2+y_z^2+z_x^2),$$

and this, with $\theta = -1$, is obviously negative provided that the shears vanish and x_x, y_y, and z_z have the same sign.

This aether is perhaps not so satisfactory as the quasi-rigid one, because the corresponding theory is limited to isotropic and homogeneous media, so that also the case of boundary surfaces has to be left out of account.

15. Graetz's Theory

The last mechanical aether theory to be still considered is due to Graetz. In this the second set of equations is obtained in the same way as with the quasi-labile aether. In fact, what in the last case was achieved by putting $\theta = -1$ in the equation of motion of an ordinary elastic medium,

$$\rho\frac{\partial^2 \xi}{\partial t^2}=K\left\{\Delta\xi+(1+2\theta)\frac{\partial P}{\partial x}\right\},$$

with P written for $\dfrac{\partial \xi}{\partial x}+\dfrac{\partial \eta}{\partial y}+\dfrac{\partial \zeta}{\partial z}$ [and Δ for $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$],

Graetz obtains by adding to the right-hand member a term $-2K(1+\theta)\partial P/\partial x$, which converts it into

$$K\left(\Delta\xi - \frac{\partial P}{\partial x}\right) = -2K\left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right).$$

His theory then amounts to retaining for the free aether the ordinary elastic equations, so that in a vacuum Maxwell's equations do not hold. After all, since the free aether cannot be experimented with, we shall never be able to find out whether these equations do hold for it or not. For a ponderable medium Graetz assumes that the aether particles are acted upon not only by the surrounding aether but also by the ponderable substance. The force due to the latter gives then the term

$$-2K(1+\theta)\partial P/\partial x.$$

Of this Graetz gives also some account, inasmuch as he deduces the said force from a pressure exerted by the ponderable matter upon the aether and represented by $p = 2K(1+\theta)P$.

16. The Charging of a Conductor according to the Elastic Aether Theories

We have just seen how it is possible, by ascribing to the aether various properties, to account for Maxwell's equations. We must come back, however, to a certain difficulty which appears in all mechanical aether theories and which presents itself when we consider a charged conductor. If we look upon the electric force as the manifestation of a velocity in the aether, then, as was already mentioned, we must imagine a continuous aether stream towards or from the conductor, though we perceive in the latter no change whatever.

How have we to picture to ourselves a charged conductor on the theory in which the magnetic force is represented by an aether stream? The dielectric displacement is then a rotation about the lines of force in the sense corresponding to that of the lines. The components of this rotation are $p = \frac{1}{2}(\partial\zeta/\partial y - \partial\eta/\partial z)$, etc., whence we see that the distribution of the rotation is always solenoidal. Let through every point of space a vector be drawn, indicating the direction of the rotation. Then the system of curves to which these vectors are tangential will be the so-called

vortex lines, and of these, owing to the solenoidal distribution, we can also construct vortex tubes.

Let us now consider a spherical conductor. During the process of charging, electricity is being communicated to it through a wire. This means that rotations are being produced in the aether, and therefore angular velocities exist while the conductor is being charged. Let us, then, consider the vortex lines in the whole space surrounding the sphere. In virtue of the solenoidal distribution these lines must either be re-entrant or extend from infinity to infinity. To each of these vortex lines must correspond another within the wire. One would now have to imagine that the aether fibres inside the wire are twisted and that this twist is propagated through the conductor and thence along the lines of force in the medium. The rotation in the latter is opposed by the elasticity of the medium giving rise to couples which, in absence of the rotating force, would at once untwist the system, that is to say, discharge the conductor. Now, the difficulty consists in finding out what happens when the wire is removed. For the state of affairs which actually takes place cannot exist in the picture just given.

In order to see this, we consider an arbitrary surface σ, whose normal at a point x, y, z has the direction angles α, β, γ, and which is bounded by a line s. Then, by Stokes' theorem,

$$\int_\sigma (p \cos \alpha + q \cos \beta + r \cos \gamma) d\sigma = \tfrac{1}{2} \int_s (\xi dx + \eta dy + \zeta dz). \quad (23)$$

The field around the charged sphere after the removal of the wire is perfectly symmetrical. Now, if we take for σ a portion of a concentric sphere, the surface integral represents, apart from a constant factor, the quantity of electricity which passed through this surface and is thus equal to the charge of the portion of the spherical conductor which is cut out by the cone subtended by s and having its vertex at the centre of the sphere. The same charge must also be represented by the line integral in (23). We begin with taking for σ a small portion of the concentric sphere and we let this slowly increase. (The successive boundary lines s may, *e.g.*, consist of a system of parallel circles.) Then the surface integral will continually increase, while this is impossible for the line integral after σ has become greater than a hemisphere. Ultimately when the boundary s dwindles to a

point, the surface integral represents the charge of the whole sphere, while the line integral becomes nil. The point is that, as is shown by (23), it is for purely kinematical reasons impossible to have at each place on the surface of the sphere a rotation around the radius which, seen from outside, has everywhere the same sense. The difficulty arises from the fact that the rotation is throughout solenoidal, while the dielectric displacement has not this property wherever there are charges.

Larmor, Reiff, and others tried to save the theory by giving up the symmetry around the sphere. They assumed that at the place where the wire originally was the conditions are somewhat different from the remainder of the sphere, viz. that there is a canal K at that place within which the aether is loosened from the surrounding aether, so that while the aether in K is fixed, the surrounding one acquires rotations corresponding to the electric force, with the result that the line integral of the aether displacement along a path embracing the canal is equal to the whole charge of the sphere. It should be possible to keep up such a state by applying to the aether outside K, all along the surface of the canal, appropriate tangential external forces. Instead of this, one might think of attaching the twisted aether to that within K, which would prevent a complete untwisting of the medium outside of K. No objection can now be derived from (23), since in this equation ξ, η, ζ are assumed to be throughout continuous, while this condition does not hold at the surface of K.

III

KELVIN'S MODEL OF THE AETHER

17. Kelvin's Model of the Quasi-rigid Aether

Kelvin conceived a model of a quasi-rigid aether built up of gyrostats. This is a complicated problem. It amounts to finding a system which permits all deformations but resists such as are associated with rotations and no others. The idea occurred to him to meet this requirement by means of gyrostats, for these oppose themselves to any change of the direction of their axes. Thus his task was to find a system containing a number of lines which remain parallel to their original direction at every deformation devoid of rotation, and which change their direction as soon as rotations are produced in the system. Along these lines one had then to lay bars bearing gyrostats.

Let us consider a homogeneous deformation in which, that is, the components of the displacement of any of the points of the system, ξ, η, ζ, are linear functions of its co-ordinates x, y, z,

$$\xi = a_1 + a_{11}x + a_{12}y + a_{13}z,$$
$$\eta = a_2 + a_{21}x + a_{22}y + a_{23}z,$$
$$\zeta = a_3 + a_{31}x + a_{32}y + a_{33}z.$$

If the coefficients a_{11}, etc., are chosen arbitrarily, the deformation thus expressed is in general associated with a rotation. This can be found geometrically by recalling that a sphere is transformed into an ellipsoid and by determining those mutually perpendicular diameters of the sphere which correspond to the axes of the ellipsoid. The algebraic representation is found by rewriting the formulae thus :

$$\xi = a_1 + a_{11}x + \tfrac{1}{2}(a_{12} - a_{21})y + \tfrac{1}{2}(a_{13} - a_{31})z + \tfrac{1}{2}(a_{12} + a_{21})y + \tfrac{1}{2}(a_{13} + a_{31})z,$$

$$\eta = a_2 + a_{22}x + \tfrac{1}{2}(a_{23} - a_{32})z + \tfrac{1}{2}(a_{21} - a_{12})x + \tfrac{1}{2}(a_{23} + a_{32})z + \tfrac{1}{2}(a_{21} + a_{12})x,$$

$$\zeta = a_3 + a_{33}z + \tfrac{1}{2}(a_{31} - a_{13})x + \tfrac{1}{2}(a_{32} - a_{23})y + \tfrac{1}{2}(a_{31} + a_{13})x + \tfrac{1}{2}(a_{32} + a_{23})y.$$

Thus the deformation is split into a shift, a dilatation, a rotation, and a shear, and we see that the conditions for an irrotational deformation are $a_{12} = a_{21}$, $a_{23} = a_{32}$, $a_{31} = a_{13}$. In such a deformation, therefore, six coefficients, apart from the shifts, still remain undetermined.

Now, to arrive at Kelvin's model, we construct in a plane a system of congruent equilateral triangles fitting to each other and erect upon these triangles as bases, omitting every second (as shown in the figure), regular tetrahedra. The corners of these tetrahedra lie again in a plane and form a system of points such as those of the ground-plane. On these points, therefore, such a system of tetrahedra can again be constructed. This we do so that the bases of these tetrahedra should have the same position as those in the first layer of tetrahedra, so that the former can be obtained from the latter by a mere shift. That is to say, if the non-shaded triangles of the figure were first chosen as bases, we now take as bases the triangles *PQR*, *RST*, etc. In this way we can proceed, and similarly the system can be indefinitely continued on the other side of the ground-plane. Thus every corner point of the system will be the common vertex of four tetrahedra, at which also twelve edges will meet, two by two being prolongations of each other. Kelvin imagines now placed at every corner point a ball from which issue six bars and as many tubes, all of these being free to assume any direction whatever. The bars of one ball are now put into the tubes of other balls wherein they can be freely shifted back and forth.

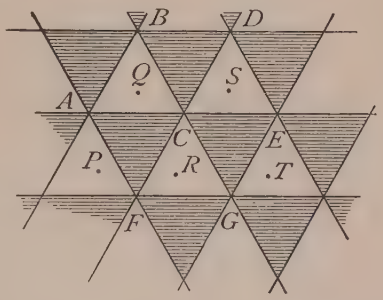

FIG. 12.

This system can now be built up in the form of tetrahedra as just described. It will not oppose itself to any pure [irrotational] deformation.

In the next place Kelvin introduces into each of our tetrahedra a system of three rigidly connected and mutually perpendicular bars of variable length (which may again be accomplished by the bar-and-tube method), and whose ends must remain in the grooves of the bars of the first system, so that they join each time two opposite edges of the tetrahedron. In a regular tetrahedron these bars coincide with the lines joining the mid-points of the opposite edges, but also in any tetrahedron whatever a set of mutually orthogonal intersecting lines, joining pairs of opposite

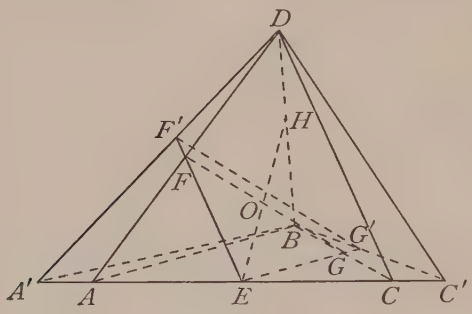

FIG. 13.

edges, can always be assigned.* The introduction of these systems of bars does not prevent any deformation. We will now prove that with an irrotational deformation of the original system the new bars are always shifted parallel to their initial direction.

We first consider the case in which a regular tetrahedron undergoes an infinitesimal dilatation in the direction of one of the edges, say AC, which leaves the plane passing through BD and E, the mid-point of AC, in its place. It is enough to prove that a set of mutually perpendicular joins of opposite

* To see this, notice that through a given point one and only one line can always be drawn which joins two skew lines. If, therefore, P be an arbitrary point within a tetrahedron, there are through P three determined lines, each of which cuts a pair of opposite edges. By requiring these lines to be mutually perpendicular we have three equations for the co-ordinates of P. Their solution for the case of a tetrahedron which differs but infinitesimally from a regular one is implied in the following considerations of the text.

edges of the new tetrahedron can be found which are parallel to the joins of the mid-points of the original tetrahedron. For this will then be the only existing set. Now, leaving the line EH unchanged, let us consider FG. If this is to remain parallel to its original direction, and still cut the line EH, its intersection point with the plane ADC must remain on the line EF and thus be shifted along it, say to F'. It is thus enough to show that the line through F' parallel to FG cuts BC' or, if G' be the point where that line cuts EG, that $FF' = GG'$. Now, this follows from the congruence of the triangles DFF' and BGG'. Similarly for the line joining the mid-points of AB and CD; in view of the symmetry, it is obvious that this line will again be shifted along EH, so that we find, in fact, three orthogonal joins of the opposite edges, parallel to the original ones. What was just proved for a dilatation in the direction of the edge AC holds, of course, for the remaining edges, and our proposition will be completely established by showing that every irrotational deformation can be obtained by the superposition of six dilatations along the edges and of a displacement of the whole system. This, however, follows directly from what precedes, since an irrotational deformation is just determined by six independently prescribed ones. Moreover, the relation between the values of the dilatations and the coefficients a_{11}, etc., in the general formulae can readily be established. In fact, if α_1, β_1, γ_1 be the direction cosines of the first edge, d_1 the dilatation in this direction, and similarly with changed suffixes for the remaining five edges, we have (apart from the shift)

$$\xi = d_1\alpha_1(x\alpha_1 + y\beta_1 + z\gamma_1) + d_2\alpha_2(x\alpha_2 + y\beta_2 + z\gamma_2) + \ldots + d_6\alpha_6(x\alpha_6 + y\beta_6 + z\gamma_6),$$

$$\eta = d_1\beta_1(x\alpha_1 + y\beta_1 + z\gamma_1) + d_2\beta_2(x\alpha_2 + y\beta_2 + z\gamma_2) + \ldots + d_6\beta_6(x\alpha_6 + y\beta_6 + z\gamma_6),$$

$$\zeta = d_1\gamma_1(x\alpha_1 + y\beta_1 + z\gamma_1) + d_2\gamma_2(x\alpha_2 + y\beta_2 + z\gamma_2) + \ldots + d_6\gamma_6(x\alpha_6 + y\beta_6 + z\gamma_6),$$

whence

$$\begin{aligned} d_1\alpha_1^2 + d_2\alpha_2^2 + \ldots + d_6\alpha_6^2 &= a_{11}, \\ d_1\beta_1^2 + d_2\beta_2^2 + \ldots + d_6\beta_6^2 &= a_{22}, \\ d_1\gamma_1^2 + d_2\gamma_2^2 + \ldots + d_6\gamma_6^2 &= a_{33}, \end{aligned}$$

$$d_1\alpha_1\beta_1 + d_2\alpha_2\beta_2 + \ldots + d_6\alpha_6\beta_6 = a_{12},$$
$$d_1\beta_1\gamma_1 + d_2\beta_2\gamma_2 + \ldots + d_6\beta_6\gamma_6 = a_{23},$$
$$d_1\gamma_1\alpha_1 + d_2\gamma_2\alpha_2 + \ldots + d_6\gamma_6\alpha_6 = a_{31},$$

wherewith the d's are determined, if the a's be given.*

If now at the middlemost bars an arrangement is made of such a kind that for rotating the bars a couple is necessary which, for an infinitesimal rotation, is proportional to the rotation and has its axis coinciding with the axis of the rotation, then there is still no force opposing an irrotational deformation of the first set of points. But if there is a rotation, a resisting couple is produced which is proportional to the rotation and whose axis will always concide with that of the rotation, provided the three middlemost bars have all the explained property. To see this we have only to consider one of the tetrahedra. Let OP, OQ, OR be the directions of the inner bars, and let us consider an infinitesimal rotation ϕ whose axis makes the angles λ, μ, ν with the bars. This can be resolved into a rotation $\phi \cos \lambda$ about OP, and $\phi \cos \mu$, $\phi \cos \nu$ about OQ and OR. In the first rotation OP remains in its place, but for the rotation of OQ a couple is necessary of moment $C\phi \cos \lambda$ and with axis along OP. Similarly for the rotation of OR, making in all $2C\phi \cos \lambda$. In quite the same way the rotation about OQ requires a couple $2C\phi \cos \mu$, directed along OQ, and that about OR a couple $2C\phi \cos \nu$. Compounding all these couples we have, as announced, the couple $2C\phi$ whose axis coincides with that of the rotation.

18. Solid Gyrostat

As a first device to make a couple necessary for changing the direction of a bar Kelvin proposed the ordinary gyrostat. The bar AB (Fig. 14) carries a fixed ring in which a second ring is mounted, free to rotate around $PQ \perp AB$. The diameter RS of

* In fact, it can be shown that the determinant of the coefficients of $d_1, \ldots d_6$ does not vanish, so that the equations are compatible with each other. In proving this the co-ordinate axes can be chosen arbitrarily. If for these the joins of the mid-points of the opposite edges are taken, one of the direction cosines for each edge is zero, while the others become $1/\sqrt{2}$ with the same or opposite signs. The absolute value of the determinant is then found to be equal $\frac{1}{4}$.

the second ring carries at the centre O a fly-wheel with RS as axis. It is clear, first of all, that this arrangement does not oppose itself to a rotation of the bar AB around the axis PQ. Kelvin mounts then on every bar two such rings in mutually perpendicular planes. But let us first consider a single gyrostat. Its inner ring can spin about PQ, and the fly-wheel about RS. For every contemplated rotation and similarly for every moment with respect to an axis a certain sense will be assumed as the positive one, viz. that from which the rotation appears to be anticlockwise. Let for the rotations in question the positive sense be given by OP and OR. Let ω be the angular velocity of the fly-wheel with respect to the inner ring and let $\angle AOR = \theta$, and, therefore, $\dot{\theta}$ the angular velocity about OP. Further, let Q be the moment of inertia of the fly-wheel corresponding to the axis RS, and Q' that corresponding to the axis PQ. Suppose now that, while APB is kept fixed, the inner ring and the fly-wheel are spinning. Then $Q'\dot{\theta}$ will be the moment of momentum of the fly-wheel along [about] OP, and its time-rate of change $Q'\ddot{\theta}$. This change of the moment of momentum can be resolved (Fig. 15) along OR and OT perpendicular to OR and OP, and therefore in the plane AOR. Instead of the angular velocity ω and the moment of momentum $Q\omega$ about the axis OR, the fly-wheel has after a time dt the angular velocity $\omega + d\omega$ and the moment of momentum $Q(\omega + d\omega)$ about the axis OR' ($\angle ROR' = d\theta$). Thus the components of the moment of momentum are, up to terms of the second order, $Q(\omega + d\omega)$ along OR and $Q\omega d\vartheta$ along OT, and its rate of change $Q\dot{\omega}$ about OR and $Q\omega\dot{\theta}$ about OT. The couples produced by this change of the moment of momentum are due to the forces exerted by the inner ring on R and S which, however, can give no couple about OR. Thus, $Q\dot{\omega} = 0$, whence

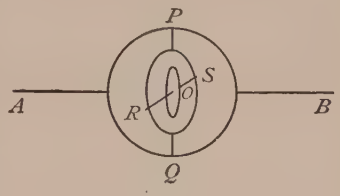

Fig. 14.

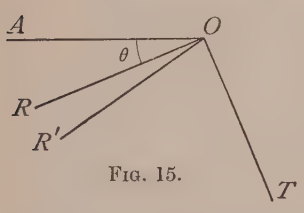

Fig. 15.

$$\omega = \text{constant}.$$

Let us now consider the system consisting of the inner ring and the fly-wheel. This is acted upon by forces applied at P and Q, which thus are unable to produce a couple about OP. If now q be the moment of inertia of the inner ring with respect to the axis OP, the moment of momentum of the whole system relatively to this axis is $Q'\dot{\theta}+q\dot{\theta}$, and since this cannot vary, $\dot{\theta}$ itself is constant.

Let us now see what happens when the bar carrying the gyrostat is turned from its position A_0B_0 (Fig. 16) to AB contained in the plane $A_0P_0B_0$ and making with A_0B_0 an infinitesimal angle ϵ. We will suppose that the bar is being kept in its new position and that the exterior ring is held in the original plane $A_0P_0B_0$, so that OP lies in this plane perpendicularly to AB. The forces which through this change of position are brought to act upon the inner ring have their points of application at P and Q and can thus produce no couple about PQ. Let there be initially no rotation about OP_0 and let RS fall initially into the direction of $A_0B_0(\theta=0)$. Then we shall have only the moment of momentum about OA_0 amounting to $Q\omega$, which we resolve into $Q\omega$ along OA and $Q\omega\epsilon$ along OP. Now, suppose that owing to the said change of position the inner ring which originally did not spin acquires an angular velocity $\dot{\theta}$. (This will turn out presently to have a non-vanishing value.) This angular velocity is shared also by the fly-wheel, but the moment of its momentum about the axis OR does not depend on the angular velocity $\dot{\theta}$ but only on the angular velocity around OR, and the latter must thus still retain its original value ω, because during the considered change of position the fly-wheel was acted upon only by such forces, at R and S, whose moment with respect to OR is nil. Thus, after the displacement, we have the moments of momenta $Q\omega$ along OA and $(Q'+q)\dot{\theta}$ along OP, and since there is no couple about OP,

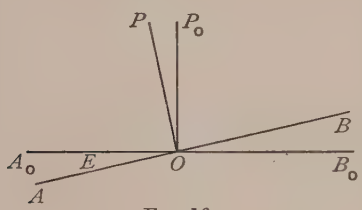

FIG. 16.

$$Q\omega\epsilon=(Q'+q)\dot{\theta},$$

whence

$$\dot{\theta}=\frac{Q\omega\epsilon}{Q'+q}.$$

Through the change of position of the bar the innermost ring is thus set spinning, the direction and the velocity of this spin being determined by the last formula. The subsequent course of things can be found by applying the considerations of the early part of this article (since these hold for every position of the bar AB). We can thus conclude that the acquired angular velocity $\dot{\theta}$ will remain without change of size, so that the angle θ will gradually mount to considerable values, and that the moment of momentum of the fly-wheel will undergo, per unit time, a change of which the component along OT, perpendicular to the plane of the inner ring, is $Q\omega\dot{\theta}$. An equal couple is necessary to keep APB in the new position, and substituting for $\dot{\theta}$ the value just found, this couple turns out to be

$$\frac{Q^2\omega^2}{Q'+q}\epsilon$$

and is directed along OT. Resolving it into components along OA and $ON \perp OA$ (corresponding to $\theta = 90°$), we have, along OA,

$$-\frac{Q^2\omega^2}{Q'+q}\epsilon \sin\theta,$$

and along ON,

$$\frac{Q^2\omega^2}{Q'+q}\epsilon \cos\theta.$$

It is the latter component which is required for our theory. In fact, this couple whose direction coincides with that of the rotation from A_0B_0 to AB is necessary to keep the bar in the new position, in other words, the bar resists this rotation with an equal couple of the opposite sense. The other component, proportional to sin θ, is due to the circumstance that some forces are also necessary to keep the plane $ABPQ$ in its original position. To provide for this we can mount upon our axis yet another gyrostat in the same plane with and entirely similar to the first, with the only difference that its fly-wheel spins originally in the opposite sense. Then $\dot{\theta}$ will have for the two gyrostats always opposite values, and since the initial value of θ is 0 for both, their angles θ themselves will be equal and of opposite signs. The couple about ON, necessary to reset the bar, will thus, of course, be twice as great. Similarly, the gyrostat which we have already introduced in the plane perpendicular to that of the first can be replaced by a set of two oppositely spinning gyrostats,

so that ultimately each of the bars carries four gyrostats. Then the system resists every motion associated with a change of the direction of the bars and no other, exactly as was required. The serious objection against this arrangement (as an aether model) is that the couple required to keep the bar in the new position becomes, owing to the factor $\cos\theta$, smaller and smaller (and finally even negative).

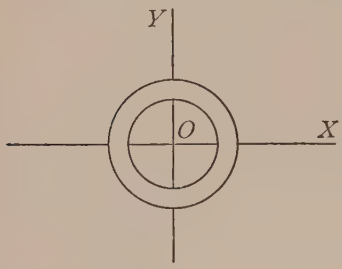

FIG. 17.

19. LIQUID GYROSTAT

Yet another device was proposed by Kelvin to ensure that a couple should be necessary for changing the direction of the bars, viz. a *liquid gyrostat*. This consists of a ring-shaped tube (Fig. 17) filled with a circulating liquid and free to rotate about one of its diameters as axis. This axis lies in one of our previous bars. Such a single ring has the same effect as the previous solid gyrostat.

The theory is much the same as before. Let the axis about which the ring rotates be chosen as X-axis and let the position of the ring be determined by the angle θ contained between the normal of its plane, taken in the sense appropriate to the motion of the liquid, and the axis OZ. For $\theta=0$ the ring would then have the position shown in Fig. 17, where the Z-axis is assumed to point forward. The actual position of the ring at any instant follows therefrom by a rotation θ about OX. Compare Fig. 18, where OZ' is the normal to the plane of the ring, while OY' is contained in that plane.

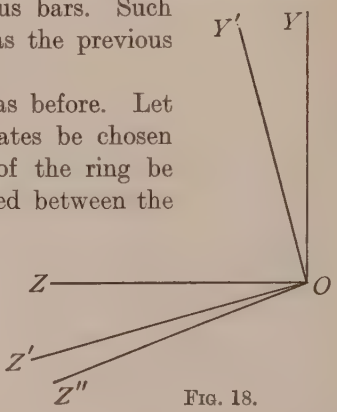

FIG. 18.

Let the angular velocity of the liquid be ω and the moment of inertia of the liquid with respect to the axis of the ring Q, and that with respect to a diameter Q' (so that, the tube being very slender, $Q'=\frac{1}{2}Q$).

Consider first the liquid itself. This has at first the moment of momentum $Q'\dot{\theta}$ about OX and $Q\omega$ about OZ', and then $Q'(\dot{\theta}+d\dot{\theta})$ about OX and $Q(\omega+d\omega)$ about OZ''. The vectorial difference between the moment $Q(\omega+d\omega)$ about OZ'' and $Q\omega$ about OZ' can be resolved along OY' and OZ', giving as components of the rate of change of the moment of momentum $Q\dot{\omega}$ along OZ' and $-Q\omega\dot{\theta}$ along OY'. The liquid is subjected to no other forces than the pressure of the tube walls, and in virtue of a known property of surfaces of revolution that pressure gives rise to a system of forces all of which cut the axis OZ' and thus can give no couple about OZ'. Consequently, $Q\dot{\omega}=0$, and therefore $\omega=$constant. To consider now the system made up of the liquid and the tube, let the moment of inertia of the latter relatively to the axis about which it can spin be q. Then the rate of change of the moment of momentum of the system along OX will be $(Q'+q)\ddot{\theta}$, and since the system is acted upon only by forces exerted by the axle, this is again zero, so that $\dot{\theta}$ is constant. Thus, while the axis OX is kept fixed, the ring can spin about it uniformly and at the same time the liquid can circulate in the tube with constant velocity. But to keep the axis in position we must apply to it a couple $-Q\omega\dot{\theta}$ about the line OY', for, as we saw, the latter is the rate of change of the moment of momentum of the liquid.

Suppose now that initially $\theta=0$ and $\dot{\theta}=0$, so that we have to do with the case represented in Fig. 17. No work, of course, is required to turn around the bar in the plane XOY, but a rotation about OY is opposed by a couple. To see this, we consider a rotation about OY through an infinitesimal angle ϵ, which brings the bar OX (Fig. 19) into the position OX' in which it is again kept fixed. The normal ON of the plane of the ring has now moved into the plane YOZ' and will first coincide with OZ' or at the utmost include with this direction an angle of the order of ϵ; it has acquired, however, an angular velocity $\dot{\theta}$ about OX' which will differ from zero though it can only be small of the order of ϵ (since it is the effect of a change of position determined by ϵ). In fact, after the readjustment we have the moment of momentum $(Q'+q)\dot{\theta}$ about OX' and $Q\omega'$ about ON, and resolving both along OX and OZ and keeping in mind that initially θ and $\dot{\theta}$ vanished, we find for the increment along OX

the value $(Q'+q)\dot{\theta}+Q\omega\epsilon$, and as for obvious reasons this must vanish, we have

$$\dot{\theta}=-\frac{Q\omega}{Q'+q}\epsilon.$$

Let us still notice that resolving the moment of momentum of the liquid after the rotation ϵ along the axes OX, OY, OZ, the value of the last of these is found to be $Q\omega'$, since terms of the order ϵ^2 can be neglected. And since the forces exerted on the liquid during the change of the direction of the bar give no moment with respect to OZ, ω' must be equal to ω.

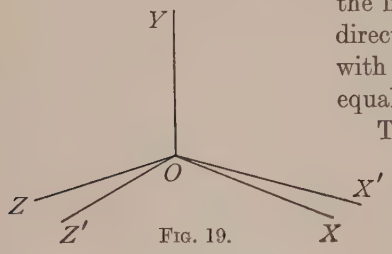

Fig. 19.

The value found for $\dot{\theta}$ is the angular velocity acquired by the ring through the change of position of the bar, with which it continues to spin. For the subsequent motion holds all that was just said about the possible motions in the original position of the bar. If OY and OZ in Fig. 18 are taken to represent the directions denoted in Fig. 19 by OY and OZ', it will become manifest that the bar must be acted upon by a couple about OY' amounting to

$$-Q\omega\dot{\theta}=\frac{Q^2\omega^2}{Q'+q}\epsilon,$$

which can again be resolved into $\frac{Q^2\omega^2}{Q'+q}\epsilon\cos\theta$ along OY and $\frac{Q^2\omega^2}{Q'+q}\epsilon\sin\theta$ along OZ. The rotation of the bar about OY requires thus first of all a couple about OY proportional to the rotation. But in addition to this another couple, about OZ, is required which can again be avoided by mounting upon the axle two gyrostats with liquids circulating in opposite senses. Then, of course, the required couple will again be doubled, *i.e.* amount to

$$2\frac{Q^2\omega^2}{Q'+q}\epsilon\cos\theta.$$

The difficulty pointed out in connection with the solid gyrostats exists also in the present case. These models can therefore be

used only if the bar carrying the gyrostat is not being displaced too long in the same direction, as *e.g.* in the case of periodical motions of small amplitude.

We will consider the case in which ϵ is a periodic function of the time (and remains very small). Then the same property must also hold for the couple determining the position of the bar. This vibration about the axis OY is then associated with an oscillation of the ring-shaped tube around the bar. In other words, also θ will be a periodic function of the time and the deviations will remain small. Moreover, since the position changes of the bar are very small, these oscillations of the tube can be considered as taking place about the axis OX (Fig. 19). Thus we have the following moments of momenta: $(Q'+q)\dot{\theta}$ about OX, $(Q'+q)\dot{\epsilon}$ about OY, and $Q\omega$ about the axis ON normal to the plane of the ring. This axis changes continually its position and makes at a given instant the angles ϵ, $-\theta$, 0 with the axes OX, OY, OZ. In considering the moment $Q\omega$ about ON, the latter axis cannot be replaced by OZ, for though these directions differ but very little, yet the moment of momentum itself is not small. This, therefore, must still be resolved along OX, OY, and OZ. The component along OX is $Q\omega\epsilon$, that along OY, $-Q\omega\theta$, and that along OZ, $Q\omega$. Let us now suppose there were no other external forces than the couple producing the oscillations of the bar in the plane XOZ, which therefore has OY for its axis, and let the moment of this couple be K which, of course, is a periodic function of the time. We shall then have the following equations of motion: first, by considering the moment of momentum about OX,

$$Q\omega\epsilon + (Q'+q)\dot{\theta} = 0,$$

the zero on the right hand being conditioned by the vanishing initial values of ϵ and $\dot{\theta}$, and, second, by considering the moment of momentum about OY,

$$-Q\omega\dot{\theta} + (Q'+q)\ddot{\epsilon} = K,$$

while the third equation of motion expresses simply that ω is constant. Eliminating $\dot{\theta}$,

$$\frac{Q^2\omega^2}{Q'+q}\epsilon + (Q'+q)\ddot{\epsilon} = K.$$

This has the form of the equation of forced vibrations. We can also speak of proper or free vibrations of our system, which may exist when $K=0$. The frequency of these free vibrations is

$$\frac{Q\omega}{Q'+q}$$

and their period

$$\frac{2\pi}{\omega}\frac{Q'+q}{Q}.$$

In presence of an external force whose period is large compared with that of the free vibrations, the state at any instant will coincide with that in which the system would be in equilibrium under the action of the force prevailing at that instant. If the period of the external force is small, the phase difference between the vibrations of the system and of the force is $\frac{1}{2}\pi$. If the liquid is assumed to circulate in the tubes very rapidly, we may always limit ourselves to the first case. Mounting then on each of the inner bars four gyrostats, in a manner already explained, we can represent the aether quite well. This can also be so arranged (by varying Q or ω) as to yield a non-homogeneous medium, by means of which also the refraction and reflection phenomena can be represented. For the free aether the period of the external couple K must then always be assumed very long as compared with that of the free vibrations, and thus also with the time of revolution of the liquid in the tubes. Otherwise the term with $\ddot{\epsilon}$ in our equation would come into prominence, and the aether would have to be given different densities for vibrations of different frequencies, which in turn would influence the velocity of propagation.

20. Liquid in Turbulent Motion as Aether Model

Kelvin tried also to represent the aether by means of an incompressible liquid in turbulent motion. The dimensions characterising this medium are those of the vortices. Upon these a coarser motion can be superposed, as *e.g.* a propagation along the Y-axis of transversal vibrations in the XOY-plane. If $f(y,t)$ be the velocity of a particle due to this vibration [along the X-axis] and u', v', w' the velocity

components of the turbulent motion at the point in question, the resultant velocity is $u=f(y,t)+u'$, $v=v'$, $w=w'$. The velocity corresponding to the vibration can be obtained separately by averaging over spaces whose dimensions are small compared with the wave-length of the oscillatory motion but large compared with the vortices. The average velocity is then nil for the turbulent motion, while $f(y,t)$ is nearly equal throughout the domain over which we have averaged. The averages thus defined will be expressed by bars over the letters representing the magnitudes in question. We will now deduce two equations which will show that transversal vibrations can, in fact, propagate themselves and will give us also their propagation velocity. The first of these is arrived at by considering the momentum along the X-axis carried across a plane perpendicular to the Y-axis. In fact, such a plane is traversed by the liquid in either direction and even in equal amounts, but the liquid flowing in one direction can have a different momentum from that streaming in the opposite direction. For the excess of momentum transferred towards the positive over that in the negative direction we find, per unit area, $\rho\,\overline{u'v'}$, if ρ be the density of the liquid.* By considering the increase of momentum within a short cylinder having its faces perpendicular to the Y-axis, we find as the first of the required equations

$$\frac{\partial f(y,t)}{\partial t} = -\frac{\partial(\overline{u'v'})}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Next, we have the usual equations of motion of a non-viscous liquid,

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} \\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} \end{aligned}\right\}, \quad . \quad . \quad . \quad (25)$$

where p is the pressure divided by the density.

* This can be compared with the momentum transfer in a gas due to the passage of molecules through a plane, as considered in the theory of viscosity.

Substituting our special values, $u = u' + f(y,t)$, etc., the first two equations become

$$\frac{\partial f(y,t)}{\partial t} + \frac{\partial u'}{\partial t} = -\left\{ f(y,t)\frac{\partial u'}{\partial x} + v'\frac{\partial f(y,t)}{\partial y} \right\} - \left\{ u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} + \frac{\partial p}{\partial x} \right\},$$

$$\frac{\partial v'}{\partial t} = -f(y,t)\frac{\partial v'}{\partial x} - \left\{ u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z} + \frac{\partial p}{\partial y} \right\},$$

whence

$$v'\frac{\partial f(y,t)}{\partial t} + \frac{\partial (u'v')}{\partial t} = -\left\{ f\frac{\partial (u'v')}{\partial x} + v'^2\frac{\partial f}{\partial y} \right\} - \left\{ u'\frac{\partial (u'v')}{\partial x} + v'\frac{\partial (u'v')}{\partial y} + w'\frac{\partial (u'v')}{\partial z} + v'\frac{\partial p}{\partial x} + u'\frac{\partial p}{\partial y} \right\}.$$

Averaging both sides, the equation can be considerably simplified. Both $f(y,t)$ and $\partial f(y,t)/\partial t$ can be considered as constant over the averaging space. Thus,

$$\overline{v'\frac{\partial f(y,t)}{\partial t}} = \overline{v'}\frac{\partial f(y,t)}{\partial t} = 0,$$

since $\overline{v'} = 0$. Again we have $\frac{\partial \overline{\phi}}{\partial x} = 0$ for every magnitude ϕ which fluctuates over distances small compared with the dimensions of the averaging space. We have still to take care of terms such as $u'\partial(u'v')/\partial x$. If this be written

$$u'\frac{\partial (u'v')}{\partial x} = \frac{\partial u'(u'v')}{\partial x} - u'v'\frac{\partial u'}{\partial x},$$

it will be seen that in the process of averaging we are left with the average of the expression

$$-u'v'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right),$$

but this is everywhere zero [the liquid being incompressible]. Thus our equation becomes

$$\frac{\partial \overline{(u'v')}}{\partial t} = -\overline{v'^2}\frac{\partial f(y,t)}{\partial} - \left(\overline{v'\frac{\partial p}{\partial x}} + \overline{u'\frac{\partial p}{\partial y}}\right). \quad . \quad . \quad (26)$$

Now, let $p_0\rho$ be the pressure when $f(y,t)$ is throughout zero, and $p_0\rho + p'\rho$ the actual pressure, and let us assume that $f(y,t)$ is always very small in comparison with the velocities of the turbulent motion. In absence of vibrations the left-hand member of (26) and the first term of the right-hand member are zero, so that

$$\overline{v'\frac{\partial p_0}{\partial x}} + \overline{u'\frac{\partial p_0}{\partial y}} = 0.$$

We will assume that this equation holds also when u' and v' are taken to stand for the velocities of the turbulent motion accompanying the vibratory motion. Then we can put

$$\overline{v'\frac{\partial p}{\partial x}} + \overline{u'\frac{\partial p}{\partial y}} = \overline{v'\frac{\partial p'}{\partial x}} + \overline{u'\frac{\partial p'}{\partial y}}. \quad . \quad . \quad . \quad (27)$$

A weak point of this theory, however, is, among others, that one does not know whether the properties of the turbulent motion remain unaltered by the vibrations. But this and similar difficulties need not detain us, our only purpose being to describe Kelvin's scheme in its main lines.

In order to transform (27) we return to the equations

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} \end{aligned}\right\}, \quad . \quad . \quad . \quad (25)$$

from which follows:

$$\Delta p = -\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} - 2\left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right).$$

If now u, v, w are replaced by the values they have in the presence of vibrations, $f(y,t)$ affects only the term with $\partial u/\partial y$, and thus we find

$$\Delta p' = -2\frac{\partial f(y,t)}{\partial y}\frac{\partial v'}{\partial x}.$$

The solution of this equation can be written symbolically

$$p' = -2\Delta^{-1}\frac{\partial f(y,t)}{\partial y}\frac{\partial v'}{\partial x}.$$

The meaning of the symbol Δ^{-1} is known from the theory of Poisson's equation $\Delta\phi = \chi$, whose solution is

$$\phi = \Delta^{-1}\chi = -\frac{1}{4\pi}\int\frac{\chi}{r}d\tau.$$

In our case, owing to the rapid changes of sign of $\partial v'/\partial x$, while $\partial f(y,t)/\partial y$ varies much slower, only the nearest neighbourhood of the point for which p' is to be evaluated contributes to the space integral. Consequently, $\partial f(y,t)/\partial y$ can be regarded as a constant factor, so that

$$p' = -2\frac{\partial f(y,t)}{\partial y}\Delta^{-1}\frac{\partial v'}{\partial x}.$$

Putting

$$v'\frac{\partial p'}{\partial x} + u'\frac{\partial p'}{\partial y} = Q,$$

and keeping in mind that $\partial f(y,t)/\partial t$ varies slowly, we find

$$Q = -2\frac{\partial f(y,t)}{\partial y}\left(v'\frac{\partial}{\partial x} + u'\frac{\partial}{\partial y}\right)\Delta^{-1}\frac{\partial v'}{\partial x}$$

$$= -2\frac{\partial f(y,t)}{\partial y}\left(v'\frac{\partial^2}{\partial x^2} + u'\frac{\partial^2}{\partial x\partial y}\right)\Delta^{-1}v'. \quad . \quad . \quad (28)$$

Now, by averaging,

$$\overline{v'\frac{\partial^2}{\partial x^2}\Delta^{-1}v'} = \overline{v'\frac{\partial^2}{\partial z^2}\Delta^{-1}v'},$$

and, though this is not quite correct, we may assume that also the third mean,

$$\overline{v'\frac{\partial^2}{\partial y^2}\Delta^{-1}v'},$$

has the same value. Thus,

$$\overline{v'\frac{\partial^2}{\partial x^2}\Delta^{-1}v'} = \tfrac{1}{3}\overline{v'\Delta\Delta^{-1}v'} = \tfrac{1}{3}\overline{v'^2}.$$

To find the last term of (28), we put $\frac{\partial}{\partial y}\Delta^{-1}v' = \psi$. Then

$$u'\frac{\partial\psi}{\partial x} = \frac{\partial(u'\psi)}{\partial x} - \psi\frac{\partial u'}{\partial x}.$$

In averaging, the first term contributes nothing, so that

$$\overline{u'\frac{\partial\psi}{\partial x}} = \overline{w'\frac{\partial\psi}{\partial z}} = -\tfrac{1}{2}\overline{\psi\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right)} = \tfrac{1}{2}\overline{\psi\frac{\partial v'}{\partial y}},$$

$$\overline{Q} = -2\frac{\partial f(y,t)}{\partial y}\overline{\left\{\tfrac{1}{3}v'^2+\tfrac{1}{2}\frac{\partial v'}{\partial y}\frac{\partial}{\partial y}(\Delta^{-1}v')\right\}},$$

and by reductions similar to those just made,

$$\overline{Q} = -2\frac{\partial f(y,t)}{\partial y}\left\{\tfrac{1}{3}\overline{v'^2}-\tfrac{1}{2}\overline{v'\frac{\partial^2}{\partial y^2}\Delta^{-1}v'}\right\}$$

$$= -2\frac{\partial f(y,t)}{\partial y}\left\{\tfrac{1}{3}\overline{v'^2}-\tfrac{1}{6}\overline{v'\Delta\Delta^{-1}v'}\right\},$$

i.e.

$$\overline{Q} = -\tfrac{1}{3}\overline{v'^2}\frac{\partial f(y,t)}{\partial y}.$$

If R^2 be the mean square of the velocity in the turbulent motion, $\overline{u'^2}+\overline{v'^2}+\overline{w'^2}=3\overline{v'^2}=R^2$, and (26) becomes

$$\frac{\partial(\overline{u'v'})}{\partial t} = -\tfrac{2}{9}R^2\frac{\partial f(y,t)}{\partial y}. \quad . \quad . \quad . \quad (29)$$

This is the second of the required equations. Combining it with our first-found equation,

$$\frac{\partial(\overline{u'v'})}{\partial y} = -\frac{\partial f(y,t)}{\partial t}, \quad . \quad . \quad . \quad . \quad (30)$$

let us eliminate $\overline{u'v'}$. This gives

$$\frac{\partial^2 f}{\partial t^2} = \tfrac{2}{9}R^2\frac{\partial^2 f}{\partial y^2},$$

which shows that in our liquid transversal vibrations can be propagated, with the velocity $\frac{\sqrt{2}}{3}R$.

IV

ATTRACTION AND REPULSION OF PULSATING SPHERES

21. Nature of the Problem

Consider an incompressible frictionless liquid in which spheres are moving about. This motion may be either a translation or a swelling and shrinking of the spheres, both of which may be periodic.

We will first take up the problem of finding the motion of the liquid if that of the spheres be given.

Particles of a non-viscous incompressible liquid which once do not rotate will never rotate. There exists then a *velocity potential*, and the velocity components can be represented by

$$u=\frac{\partial\Phi}{\partial x},\ v=\frac{\partial\Phi}{\partial y},\ w=\frac{\partial\Phi}{\partial z}.$$

Owing to the incompressibility Φ will satisfy Laplace's equation $\Delta\Phi=0$. Whence it follows that the velocity at every point of a space-region will be determined, if at every point of its boundary the normal velocity is known. For then $\partial\Phi/\partial n$ all over the boundary is known, and thus also Φ throughout that space-region is determined, since it is continuous and satisfies Laplace's equation. Let us consider the space limited by a fixed closed surface at infinite distance and by the surfaces of all the spheres. Then the normal velocity all over these boundaries is known, the motion of the spheres being given. If a, b, c be the co-ordinates of the centre of one of the spheres and R its radius, the motion is determined by $\dot{a}$, $\dot{b}$, $\dot{c}$, and $\dot{R}$. It remains to find a suitable solution of $\Delta\Phi=0$. This will be supplied by

$$V_k=(-1)^k\frac{1}{k\,!}\frac{\partial^k}{\partial h_1\partial h_2\ \ldots\ \partial h_k}\left(\frac{1}{r}\right),$$

where h_1, h_2, . . . h_k are measured along arbitrarily chosen directions [and r is the distance from the centre of the sphere]. This is a homogeneous function of the space co-ordinates of degree $-(k+1)$. If we put

$$V_k = Y_k r^{-(k+1)},$$

then (the spherical harmonic function) Y_k is a homogeneous function of the zeroth degree. Next, write

$$H_k = V_k r^{2k+1} = Y_k r^k ;$$

then H_k will be a homogeneous function of the k^{th} degree which can be represented by a whole algebraic function of the co-ordinates. For at every differentiation the highest exponent of r in the denominator is increased by 2, so that the highest exponent in V_k is $2k+1$, and the remaining ones are by an even number lower. Thus in H_k all denominators fall out, while in the numerators only even powers of r occur. H_k also satisfies the equation of Laplace, but it cannot be used for representing the velocity potential in a liquid extending to infinity, since it becomes infinite at an infinite distance. On the other hand, V_k cannot be used if the origin of r lies within the liquid.

22. A Single Moving Sphere

Consider the space outside a single sphere. Then the simplest solutions are, for $k=0$, $V_0=1/r$, $Y_0=1$, $H_0=1$. If we take $\Phi=C/r$, the velocity will be radial and equal to $-C/r^2$. By an appropriate choice of C as a function of the time this solution can be adapted to a *sphere with fixed centre and variable radius* R. In fact, since at the surface $\dot{R}=-C/R^2$, we have

$$\Phi = -\frac{\dot{R}R^2}{r^2}.$$

For $k=1$, the next simple case, we find, with the X-axis along the direction of h,

$$V_1 = -\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{x-a}{r^3},\quad Y_1 = \frac{x-a}{r},\quad H_1 = x-a.$$

If we take for the velocity potential outside the sphere

$$\Phi = \frac{C(x-a)}{r^3} = \frac{C(x-a)}{r}\,\frac{1}{r^2},$$

then, at its surface,

$$\frac{\partial \Phi}{\partial n} = -2C\frac{x-a}{R}\frac{1}{R^3} = -\frac{2C\cos\lambda}{R^3}.$$

This can be adapted to the case in which the sphere has a translation velocity along the X-axis. For then the component of this velocity along the surface normal is also proportional to $\cos\lambda$. Thus,

$$\frac{-2C\cos\lambda}{R^3} = \dot{a}\cos\lambda,$$

whence

$$C = -\tfrac{1}{2}\dot{a}R^3,$$

and

$$\Phi = -\tfrac{1}{2}\dot{a}R^3\frac{x-a}{r^3}. \quad . \quad . \quad . \quad . \quad (31)$$

Since $(x-a)/r^3$ arises from the differentiation of $1/r$, this motion of the liquid can also be obtained by a superposition of the motions due to two spheres devoid of translation but pulsating in opposite phases and placed at an infinitesimal distance from each other.

Let now the liquid be in a certain state of motion and let us ask how this motion is disturbed by the presence of a sphere. It will appear that due to the latter a certain state of motion is superposed over that already existing. We will call this the reflected motion. It will be the weaker, the farther away from the sphere.

23. A Sphere at Rest in a Liquid with a given Motion

Let us consider the case in which the disturbing sphere is at rest. We choose its centre as the origin of co-ordinates. The [irrotational] motion of the liquid being given, so also is the velocity potential Φ which can be developed around the origin into

$$\Phi = \Phi_0 + \left(x\frac{\partial\Phi}{\partial x} + y\frac{\partial\Phi}{\partial y} + z\frac{\partial\Phi}{\partial z}\right) + \tfrac{1}{2}\left(\frac{\partial^2\Phi}{\partial x^2}x^2 + \ldots\right),$$

where the values of the derivatives of Φ prevailing at the centre have to be taken. The state of motion thus expressed can be interpreted as if it were due to a superposition of different

motions whose velocity potentials are represented by the successive groups of terms. Thus, a velocity potential Φ_0 represents rest, the next group,

$$\left(x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}+z\frac{\partial\Phi}{\partial z}\right),$$

expresses a constant flow with the velocity prevailing at the centre, and so on. This series development can also be put into the form

$$\Phi=\sum_k C_k r^k Y_k.$$

The reflected motion, combined with the original one, must give all over the surface of the sphere the radial velocity 0. Let us put for the reflected motion

$$\Phi=\sum_k C_k'\frac{Y_k}{r^{k+1}},$$

and let us suppose that to each of the motions to be compounded there corresponds a reflected motion. Then we must have at the surface, for every k separately,

$$C_k k R^{k-1} Y_k - C_k'(k+1)\frac{Y_k}{R^{k+2}}=0.$$

Consequently,

$$C_k'=\frac{k}{k+1}C_k R^{2k+1},$$

and the velocity potential of the reflected motion becomes

$$\Phi=\sum\frac{k}{k+1}C_k R^{2k+1}\frac{Y_k}{r^{k+1}}.$$

24. n Spheres

Let us now turn to the problem of finding the state of motion due to the arbitrarily prescribed motion of n spheres, of which the positions, that is, as well as the sizes and velocities, are given for all times. The velocity potential Φ must then satisfy Laplace's equation and the condition that at each of the surfaces $\partial\Phi/\partial n$ should have given values. We shall find Φ by superposing n solutions, each corresponding to the motion of a single sphere. Each of these solutions will be found by proceeding first as if the sphere in question existed alone and then by taking

account, in an obvious way, of the repeated reflections of the state of motion produced by that sphere. Suppose, for instance, there were only two spheres, the first of which has a given motion, while the second is at rest. Let, in terms of the velocity potential, Φ_{I} be the motion due to the presence of the first sphere alone, Φ_{II} the motion arising through the reflection of Φ_{I} at the second sphere, Φ_{III} the motion produced by the reflection of Φ_{II} at the first sphere supposed to be at rest in its instantaneous position, and so on. Then the actual motion will be found to be represented by the infinite series

$$\Phi = \Phi_{\mathrm{I}} + \Phi_{\mathrm{II}} + \Phi_{\mathrm{III}} + \ldots$$

which will converge the more rapidly the smaller the spheres in comparison with their mutual distance. This series gives the solution of the problem. For the equation of Laplace is satisfied, since each of the terms does satisfy it and the derivatives of the series can be found by differentiating separately its terms, and all boundary conditions are satisfied. In fact, at the surface of either sphere $\partial\Phi/\partial n$ is expressed by a series whose terms, as far as the second sphere is concerned, cancel each other in pairs, while for the first sphere the first term only survives and this has the value which is prescribed for its surface.

25. Two Spheres

We will now determine the forces which two pulsating spheres exert upon each other. The equations of motion

$$-\frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right), \text{ etc.}$$

become, in terms of the velocity potential,

$$\frac{\partial p}{\partial x} = -\rho\frac{\partial}{\partial x}\left[\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}\left\{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right\}\right], \text{ etc.,}$$

and since ρ is constant, these give, with V written for the velocity,

$$p = -\rho\left(\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}V^2\right) + C, \qquad . \quad . \quad . \quad (32)$$

where C is a constant. This, however, can be omitted, since a pressure which is constant all over a surface cannot produce any motion.

Let R_1 and R_2 be the radii of the spheres and l the distance of their centres. Then the velocity potential due to the motion of the first sphere is, by Art. 22,

$$\Phi = -R_1^2 \dot{R}_1 \frac{1}{r'},$$

where r' is the distance from the centre of this sphere.

If now the origin of co-ordinates be placed into the centre of the second sphere and if the X-axis be laid along the produced join of the centres, the last expression can be developed around the second sphere into the series

$$\Phi = -R_1^2 \dot{R}_1 \left[\frac{1}{l} - \frac{x}{l^2} + \ldots \right],$$

of which it will be enough to retain the first two terms, as according to our assumption the radii of the spheres are small compared with l. Since the first term is constant, the motion is determined by the second term. At the second sphere a reflected state of motion is produced, for which the velocity potential is, by Art. 23,

$$\tfrac{1}{2} R_1^2 \dot{R}_1 \frac{1}{l^2} R_2^3 \frac{x}{r^3},$$

where r denotes the distance from the centre of the second sphere. Rigorously we should take account also of the motion arising from this through a reflection at the first sphere, and so on. But since these reflected motions become rapidly weaker, they may be disregarded. The velocity potential due to the motion of the second sphere is $-R_2^2 \dot{R}_2 / r$ which, to the desired degree of accuracy, need not be supplemented by any reflected motion at all. For our purpose is to determine the force experienced by the second sphere. This, however, requires only the knowledge of the motion of the liquid in the neighbourhood of the second sphere, and the reflection of the last-mentioned motion from the first sphere would contribute only an expression containing a factor of the order $1/l^2$. Let us still put

$$\tfrac{1}{3} R_1^3 = a_1, \quad \tfrac{1}{3} R_2^3 = a_2;$$

then the whole velocity potential in the neighbourhood of the second sphere will be

$$\Phi = -\frac{1}{l}\left(1 - \frac{x}{l} - \tfrac{3}{2}\frac{a_2 x}{l r^3}\right)\frac{da_1}{dt} - \frac{1}{r}\frac{da_2}{dt}.$$

By (32), to determine the pressure, $\partial\Phi/\partial t$ is required. Now, since in differentiating with respect to the time x, etc., are kept constant,

$$\frac{\partial\Phi}{\partial t} = -\frac{1}{l}\left(1-\frac{x}{l}\right)\frac{d^2a_1}{dt^2} + \tfrac{3}{2}\frac{x}{l^2r^3}\left(a_2\frac{d^2a_1}{dt^2} + \frac{da_1}{dt}\frac{da_2}{dt}\right) - \frac{1}{r}\frac{d^2a_2}{dt^2}.$$

To find the force upon the second sphere, we have to integrate over its whole surface, but in doing so we may disregard the term $\frac{1}{2}V^2$ in (32).* Again, all terms not containing x can be omitted, since these represent a pressure uniform all over the surface which, therefore, as was just mentioned, gives rise to no force upon the sphere as a whole. We are thus left with

$$p = -\rho x\left\{\frac{1}{l^2}\frac{d^2a_1}{dt^2} + \frac{3}{2l^2R_2{}^3}\left(a_2\frac{d^2a_1}{dt^2} + \frac{da_1}{dt}\frac{da_2}{dt}\right)\right\}$$

$$= -\rho x\left\{\frac{3}{2l^2}\frac{d^2a_1}{dt^2} + \frac{1}{2l^2a_2}\frac{da_1}{dt}\frac{da_2}{dt}\right\}.$$

Now, a pressure $p = -cx\rho$ gives for the X-component of the force upon an element $d\sigma$

$$+c\rho R_2\cos^2\theta d\sigma,$$

whence, the force upon the whole sphere,

$$\int c\rho R_2\cos^2\theta d\sigma = \tfrac{4}{3}\pi c\rho R_2{}^3 = 4\pi c\rho a_2,$$

and substituting the value of c,

$$4\pi\frac{\rho}{l^2}\left(\tfrac{3}{2}a_2\frac{d^2a_1}{dt^2} + \tfrac{1}{2}\frac{da_1}{dt}\frac{da_2}{dt}\right).$$

We will suppose that the spheres pulsate rapidly, so that the perceptible force is the average of this expression over a time interval comprising many periods. Moreover, let both spheres have the same pulsation period. Total derivatives with respect to the time can then be omitted, so that the last expression for the force upon the second sphere can be written

$$\frac{4\pi\rho}{l^2}\left\{\tfrac{3}{2}\frac{d}{dt}\left(a_2\frac{da_1}{dt}\right) - \frac{da_1}{dt}\frac{da_2}{dt}\right\}, \quad . \quad . \quad . \quad (33)$$

* For, evidently, we can limit ourselves to that part of $\frac{1}{2}V^2$ which depends on the motion of the first as well as on that of the second sphere, that is to say, to the scalar product of the velocity-vector V_1, which would correspond to the motion of the second sphere alone, into V_2, due to that of the first sphere only. Now, at the surface of the second sphere V_2 is radial and V_1 tangential, so that their scalar product vanishes.

and the required average ultimately becomes

$$-4\pi\frac{\rho}{l^2}\overline{\frac{da_1}{dt}\frac{da_2}{dt}}.$$

Whence we see that, if the pulsations of the two spheres are in phase, this force is negative, that is to say, we have an attraction, and if their phases are opposite, a repulsion. Let us consider this in more detail for the case of simple harmonic oscillations. If the radius varies periodically, so also does the volume of the sphere, and with the same period, and if the changes remain small, a simple harmonic variation of the radius is associated with a simple harmonic variation of the volume. We can write, therefore,

$$a_1 = A_1 + c_1 \cos(nt + \epsilon_1),$$

where c_1 is small compared with A_1, and similarly,

$$a_2 = A_2 + c_2 \cos(nt + \epsilon_2).$$

This gives for the averaged force along the X-axis

$$K_x = -\frac{4\pi\rho}{l^2}c_1c_2n^2\,\overline{\sin(nt+\epsilon_1)\sin(nt+\epsilon_2)}.$$

In order to find the mean over a long time, it is enough to average over a full period. Now, since

$$2\sin(nt+\epsilon_1)\sin(nt+\epsilon_2) = -\cos(2nt+\epsilon_1+\epsilon_2)+\cos(\epsilon_1-\epsilon_2),$$

the required average turns out to be

$$K_x = -\frac{2\pi\rho}{l^2}c_1c_2n^2\cos(\epsilon_1-\epsilon_2).$$

Thus we have an attraction for $\epsilon_1-\epsilon_2 < \pi/2$, reaching a maximum for $\epsilon_1=\epsilon_2$, and a repulsion for $\epsilon_1-\epsilon_2 > \pi/2$, attaining a maximum for $\epsilon_1-\epsilon_2=\pi$.

26. Treatment of the Problem by means of Lagrange's Equations

The results just found might also have been deduced from Lagrange's equations of motion. In using these we will introduce

as co-ordinates, also for the case of a system of any number of spheres, the Cartesian co-ordinates a, b, c of the centres and the radii R of the spheres. The kinetic energy T is then a quadratic function of $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{R}$ with coefficients which are functions of the co-ordinates. If X be the component of the external force upon one of the spheres, taken along the X-axis, we have

$$X = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}}\right) - \frac{\partial T}{\partial a}.$$

The kinetic energy T can be split into two parts, one due to the liquid, the other due to the spheres, and in accordance with this the inner force keeping equilibrium to the force X can be divided into two parts. We then find for the force exerted upon the sphere by the liquid along the X-axis

$$\frac{\partial T_v}{\partial a} - \frac{d}{dt}\left(\frac{\partial T_v}{\partial \dot{a}}\right), \quad . \quad . \quad . \quad . \quad (34)$$

if T_v be the kinetic energy of the liquid. The latter is (owing to $\Delta\Phi = 0$)

$$T_v = \tfrac{1}{2}\rho\int\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right\}d\tau = -\tfrac{1}{2}\rho\int\Phi\frac{\partial\Phi}{\partial n}d\sigma,$$

the last integral to be extended over the surface of all the spheres (it being assumed that we need not reckon with the infinitely distant boundary), and the normal to be taken towards the liquid, *i.e.* away from the spheres (hence the negative sign). The last integral can, of course, be divided into parts, each extended over the surface of one of the spheres, thus :

$$T_v = -\tfrac{1}{2}\rho\Sigma\int\Phi\frac{\partial\Phi}{\partial n}d\sigma.$$

In the case of two pulsating spheres we have for the second sphere $\partial\Phi/\partial n = \dot{R}_2$, and therefore,

$$\int\Phi\frac{\partial\Phi}{\partial n}d\sigma = \int\left\{-\frac{1}{l}\frac{da_1}{dt}\left[1 - \frac{x}{l} - \frac{3xa_2}{2lR_2^3}\right] - \frac{1}{R_2}\frac{da_2}{dt}\right\}\dot{R}_2 d\sigma,$$

where a_1, a_2 are as explained in the preceding article. On integrating, the second and the third terms contribute nothing,

while the remaining ones are constant, and we are thus left with

$$-4\pi R_2^2\left(\frac{1}{l}\dot{R}_2\frac{da_1}{dt}+\frac{1}{R_2}\dot{R}_2\frac{da_2}{dt}\right).$$

Similarly the integral over the first sphere is

$$-4\pi R_1^2\left(\frac{1}{l}\dot{R}_1\frac{da_2}{dt}+\frac{1}{R_1}\dot{R}_1\frac{da_1}{dt}\right).$$

The symbol a appearing in (34) stands in this case for our l, and the first term of that expression gives

$$\frac{\partial T_v}{\partial l}=-\frac{4\pi\rho}{l^2}\frac{da_1}{dt}\frac{da_2}{dt}.$$

This is exactly the force we have already found. With regard to the second term in (34), notice that what we require are the forces which act on the spheres when these have only a pulsating motion, while their centres are fixed. Thus $\dot{l}=0$. In evaluating the first term of (34) we can, for obvious reasons, simplify T_v by putting $\dot{l}=0$, as we did in fact. In the second term, however, this simplification can be introduced only *after* the differentiation with respect to $\dot{l}$. The evaluation of this term, which would lead us to the first term of (33), would thus require a further consideration of T_v, into which, however, we need scarcely enter, since we are concerned only with the time derivative of $\partial T_v/\partial\dot{a}$, and the average of such a derivative over a full period is nil.

27. Pearson's Theory

Consider the case of an infinitely long period [of pulsation], that is to say, a sphere which goes on expanding for ever. Then a surface enclosing the sphere is traversed by the amount $\frac{d}{dt}\left(\frac{4\pi R_1^3}{3}\right)$ or $4\pi da_1/dt$ of fluid. Let this be denoted by e_1. Then the attraction between two such spheres will be $\rho e_1 e_2/4\pi l^2$.

Now, Pearson abolishes the spheres and supposes that there is towards certain points an incessant stream of fluid (aether sources or sinks). It is true that one cannot well picture to himself such a state of things. But as we can imagine points spread over

a surface, towards which the aether streams from the surrounding space, to spread itself then over the surface, we should also be able to place such aether sources in three-dimensional space by calling to our aid the four-dimensional space. Similarly, there would exist points (aether sinks) where the fluid is being annihilated.* Two such aether sources would then attract each other, and similarly two aether sinks, while a source and a sink would repel one another.

Apart from the strangeness of such a representation there is also another objection. In fact, if the expanding sphere be entirely omitted, there would still be a force upon the place whence the aether emanates. If a small sphere be described around the aether source, the force exerted by the aether will be a pressure upon the sphere, and one would have to imagine that the source is being displaced together with the sphere.

There is a striking difference between the present case and that of electrical actions. For here we have attraction between points [sources] of equal, and repulsion between points of opposite signs.

A similar theory was proposed by Korn to account for molecular forces and gravitation. He imagines a number of pulsating spheres, all in phase with each other; the latter coincidence is secured by enclosing the whole space in a limiting surface which is acted upon by a periodical external force. This is propagated instantaneously through the incompressible fluid and makes the volumes of all the spheres alternately increase and diminish in the same phase.

In what precedes a description was given of some of the attempts which were made in order to account for various phenomena, and especially the electromagnetic ones, by means of speculations about the structure and the properties of the aether. To a certain extent these theories were successful, but it must be admitted that they give but little satisfaction. For they become more and more artificial the more cases are required to be explained in detail. Of late the mechanical explanations of what is going on in the aether were, in fact, driven more and more to the background. For many physicists the essential

* [For the three-dimensional beings, that is.]

part of a theory consists in an exact, quantitative description of phenomena, such *e.g.* as is given us by Maxwell's equations.

But even if one adheres to this point of view, the mechanical analogies retain some of their value. They can aid us in thinking about the phenomena, and may suggest some ideas for new investigations.

SOURCES AND REFERENCES

G. G. Stokes, *On the Aberration of Light*, Phil. Mag. (3), vol. xxvii., 1845, p. 9.

H. A. Lorentz, *De aberratietheorie van Stokes*, Zittingsverslagen Kon. Akad. v. Wet. Amsterdam, i., 1892, p. 97.

H. A. Lorentz, *Over den invloed, dien de beweging der aarde op de lichtverschijnselen uitoefent*, Verslagen en Mededeelingen Kon. Akad. v. Wet. Amsterdam, (3) ii., 1886, p. 297.

H. A. Lorentz, *De l'influence du mouvement de la terre sur les phénomènes lumineux*, Archives Néerlandaises des Sciences, xxi., 1887, p. 103.

H. A. Lorentz, *De aberratietheorie van Stokes in de onderstelling van een aether, die niet overal deselfde dichtheid heeft*, Zittingsverslagen Akad. Amsterdam, vii., 1899, p. 528.

R. Reiff, *Die Fortpflanzung des Lichtes in bewegten Medien nach der elektrischen Lichttheorie*, Ann. d. Phys. u. Chem. (3) l., 1893, p. 361.

A. A. Michelson, *The Relative Motion of the Earth and the Luminiferous Aether*, Amer. Journ. of Science (3), xxii., 1881, p. 120.

A. A. Michelson and E. W. Morley, *Influence of Motion of the Medium on the Velocity of Light*, ibid. (3), xxxi., 1886, p. 377.

A. A. Michelson and E. W. Morley, *The Relative Motion of the Earth and the Luminiferous Aether*, ibid. (3), xxxiv., 1887, p. 333.

A. A. Michelson and E. W. Morley, *On the Relative Motion of the Earth and the Luminiferous Aether*, Phil. Mag. (4), vol. xxiv., 1887, p. 449.

H. A. Lorentz, *De relatieve beweging van de aarde en den aether*, Zittingsverslagen Akad. Amsterdam, i., 1892, p. 74.

J. MacCullagh, *An Essay toward a Dynamical Theory of Crystalline Reflexion and Refraction*, 1839.

W. Thomson, *Motion of a Viscous Liquid, etc.*, Math. and Phys. Papers, London, 1890, iii. art. 99.

W. Thomson, *On the Reflexion and Refraction of Light*, Phil. Mag. (4), vol. xxvi., 1888, pp. 414, 500.

G. Green, *On the Laws of Reflexion and Refraction of Light*, Cambridge Transactions, vi., 1838, p. 400.

J. Larmor, *A Dynamical Theory of the Electric and Luminiferous Medium*, London Trans., A, clxxxv., 1894, p. 719; clxxxvi., 1895, p. 695; cxc., 1897, p. 205.

J. Larmor, *Aether and Matter*, Cambridge, 1900.

R. Reiff, *Elastizität und Elektrizität*, Freiburg and Leipzig, 1891.

A. Sommerfeld, *Mechanische Darstellung der electromagnetischen Erscheinungen in ruhenden Körpern*, Ann. d. Phys. u. Chem. (3), xlvi., 1892, p. 139.

L. Boltzmann, *Über ein Medium, dessen mechanische Eigenschaften auf die von Maxwell für den Electromagnetismus aufgestellten Gleichungen führen*, ibid. (3), xlviii., 1893, p. 78.

W. Voigt, *Über Medien ohne innere Kräfte und über eine durch sie gelieferte Deutung der Maxwell-Hertzschen Gleichungen*, ibid. (3), lii., 1894, p. 665.

L. Graetz, *Über eine mechanische Darstellung der electrischen und magnetischen Erscheinungen in ruhenden Körpern*, ibid. (3), v., 1901, p. 375.
O. Lodge, *Modern Views of Electricity*, London, 1889.
W. Thomson, *On a Gyrostatic Constitution for Ether*, Math. and Phys. Papers, 1890, vol. iii. art. 100.
W. Thomson, *On the Propagation of Laminar Motion through a turbulently moving Inviscid Liquid*, Phil. Mag. (4), vol. xxiv., 1887, p. 342.
V. Bjerknes, *Vorlesungen über hydrodynamische Fernkräfte, bearbeitet nach C. A. Bjerknes' Theorie*, Leipzig, 1900.
K. Pearson, *Ether Squirts*, Amer. Journ. of Math., vol. xiii., 1891, p. 309.
A. Korn, *Eine Theorie der Gravitation und der electrischen Erscheinungen auf Grundlage der Hydrodynamik*, Berlin, 1898.

KINETICAL PROBLEMS

(1911–1912)

INTRODUCTION

In these lectures some self-contained questions concerning kinetic theories are treated. They belong partly to the domain of the kinetic theory of gases and partly to that of the electron theory. Their subject was suggested by Knudsen's investigations on very rarefied gases and by Richardson's researches on thermionic currents.

The gases offer two extreme cases which can be treated with comparative ease, one, in which the dimensions of the containing vessel are very large, and another in which these are very small compared with the mean free path of the molecules.

In the former case there will be no sliding of the gas along a solid wall and this will have the same temperature as the contiguous gas layer, while in the latter case the gas is so rarefied that the molecular collisions can be disregarded. A volume-element does then no more contain the same matter during a certain time, as can be assumed to be the case for gases of large density. The investigation of cases falling between these two extremes offers considerable difficulties. In dealing with Knudsen's investigations one can start from the second extreme case, and this is the plan which will be here adopted.

CHAPTER I

INNER FRICTION AND SLIDING, TREATED HYDRODYNAMICALLY

1. Hydrodynamical Equations of an Incompressible Viscous Liquid

To begin with, we consider an incompressible viscous liquid and write down its hydrodynamical equations. In addition to inner friction the sliding along a fixed wall will also be taken into account.*

Let u, v, w be the velocity components of the liquid, p the pressure, X_x, X_y, etc., the inner stress, including the pressure, further, μ the viscosity coefficient and, finally, ρ the density.

For the sake of clearness it may be mentioned that X_x is the X-component of the tension exerted upon a surface-element normal to the X-axis, Y_z the Y-component of the stress upon a surface-element perpendicular to the Z-axis, and so on.

In absence of external forces the equations of motion are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ (equation of continuity)}, \quad . \quad . \quad (1)$$

$$\left.\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)&=\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)&=\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)&=\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}
\end{aligned}\right\}, \quad . \quad (2)$$

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* Helmholtz and Piotrowski investigated whether there is sliding of a liquid or not. They observed the oscillations of a hollow metallic sphere filled with liquid and suspended on a twisted wire, and they found that the sliding at the metal wall was not nil. *Wiener Sitzungsber.* xl., Abt. I., 1860, p. 607.

$$\left.\begin{aligned} X_x &= -p + 2\mu\frac{\partial u}{\partial x} \\ Y_y &= -p + 2\mu\frac{\partial v}{\partial y} \\ Z_z &= -p + 2\mu\frac{\partial w}{\partial z} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (3)$$

$$\left.\begin{aligned} X_y = Y_x &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ Y_z = Z_y &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ Z_x = X_z &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned}\right\}. \quad . \quad . \quad . \quad (4)^*$$

* In deriving these formulae one considers a parallelepipedon of edges dx, dy, dz within the liquid. Multiplying both sides of each of the equations (2) by $dxdydz$, we have on the left hand the product of an element of mass into its acceleration, and on the right hand the force acting upon this liquid mass.

With regard to the equations (3) and (4) we may notice that in absence of friction $X_x = Y_y = Z_z = -p$, while the tangential stress components are all nil.

In the presence of friction all stress components will be determined by expressions depending on the manner how the velocity varies from point to point. The equations representing these connections will be linear; to a first approximation the stress components will be determined by the derivatives of the velocities with respect to the co-ordinates.

We can thus write, in general,

$$X_x = -p + a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} + \ . \ . \ . \ + a_{32}\frac{\partial w}{\partial y} + a_{33}\frac{\partial w}{\partial z},$$

$$Y_z = b_{11}\frac{\partial u}{\partial x} + b_{12}\frac{\partial u}{\partial y} + \ . \ . \ . \ + b_{32}\frac{\partial w}{\partial y} + b_{33}\frac{\partial w}{\partial z},$$

and so on.

To determine further these coefficients we take account of the symmetry relations. Imagine the liquid reflected at the plane $x=0$ and notice that for the image the same equations must hold. It will then become plain that terms having an odd number of references to the X-axis change their sign through the reflection, while those with an even number of such references retain their sign. The reference to the X-axis can occur in a fourfold way, to wit, as in ∂x, ∂u, X_y and Z_x.

Thus, using a reflection at the plane $x=0$, we shall find that X_x remains unchanged, and this gives $a_{12} = a_{21} = a_{13} = a_{31} = 0$. Again, Y_z remains what it was, giving $b_{12} = b_{21} = b_{13} = b_{31} = 0$.

Next, a reflection at the plane $y=0$ leaves X_x unchanged, whence $a_{23} = a_{32} = 0$, while Y_z changes its sign, and this gives $b_{11} = b_{22} = b_{33} = 0$.

From the fact that the exchange of the Y- with the Z-direction can have no effect upon X_x, it follows that $a_{22} = a_{33}$.

The formula for X_x thus becomes

$$X_x = -p + a_{11}\frac{\partial u}{\partial x} + a_{22}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),$$

which, in view of the equation of continuity, can be written

$$X_x = -p + (a_{11} - a_{22})\frac{\partial u}{\partial x}.$$

Taking into account that the couple resulting from the tensions upon an

With regard to the boundary conditions we may notice that, if there be no sliding, the velocity of the liquid in contact with a solid body is the same as the velocity of the latter, but if there is sliding, this equality holds only for the normal velocity components. In the latter case, therefore, the relative velocity of the outermost layer of the liquid and the solid wall has a tangential direction.

Let us now consider within the liquid near the solid wall a short cylinder whose dimensions along the normal of the wall are infinitely small compared with those in tangential directions, and let us express the condition that the forces exerted by the liquid upon this volume-element are in equilibrium with those due to the solid body.

Let h be an arbitrary direction in the tangential plane. Then the force exerted by the liquid, per unit area, in the direction h may be represented by H_n, and that exerted by the solid body by λv_h, where v_h is the relative velocity of the liquid and the solid wall and λ a proportionality factor. Thus the *boundary condition* will become

$$H_n = \lambda v_h. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If there be no sliding, then $v_h = 0$ and we must put $\lambda = \infty$.

Let the state of motion be stationary, so that $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$ all vanish, and let us further assume that the velocities are so small that their products and their derivatives can be neglected. Then, in virtue of (1), the equations of motion (2) become

$$\left.\begin{aligned} -\frac{\partial p}{\partial x} + \mu \Delta u = 0 \\ -\frac{\partial p}{\partial y} + \mu \Delta v = 0 \\ -\frac{\partial p}{\partial z} + \mu \Delta w = 0 \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad (6)$$

---

element of liquid must be zero, if we limit ourselves to magnitudes of the third order, we find $Y_z = Z_y$, whence $b_{23} = b_{32}$. Thus,

$$Y_z = b_{23}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

Again, since the liquid is isotropic, the expressions for Y_y and Z_z must follow from that for X_x by a cyclic permutation, and similarly those for Z_x and X_y from that for Y_z. Whence it follows that the coefficients in the formulae for X_x, Y_y, and Z_z must be equal, and similarly for Y_z, Z_x, and X_y.

The isotropy of the liquid implies also that if the equations be transformed to a new system of axes obtained by a rotation of the original one, the coefficients retain their values. This gives $a_{11} - a_{22} = 2b_{23} = 2\mu$.

2. Effect of Sliding upon a Liquid flowing in a Tube

As a first application we consider a liquid flowing through a narrow tube. We put the X-axis along the tube.

The equations of motion can be satisfied by putting $v=w=0$. This amounts to disregarding small lateral motions near the ends. The equation of continuity (1) then calls for $\partial u/\partial x=0$, so that u and therefore also Δu become independent of x, and, by the first of (6), the same is true of $\partial p/\partial x$. By the remaining two equations (6) the pressure must also be the same all over the cross-section. This will also be the case of $\partial p/\partial x$. Consequently the pressure varies uniformly along the tube and depends but on a single co-ordinate, x. If p_1 and p_2 be the values of p at the beginning and the end of the tube, and if l be its length, we have

$$\frac{\partial p}{\partial x}=\frac{p_2-p_1}{l}$$

and, by (6),

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{p_2-p_1}{\mu l}.$$

Limiting ourselves to a tube of circular section, of radius R, transforming to polar co-ordinates, and noticing that, since the motion can be assumed to be axially symmetrical, u depends only on r, we can write the last equation

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}=\frac{p_2-p_1}{\mu l},$$

whence, by integration,

$$r\frac{du}{dr}=\frac{1}{2}\frac{p_2-p_1}{\mu l}r^2,$$

an additive integration constant being omitted, since the velocity of flow is a maximum at the middle of the tube, so that $du/dr=0$ for $r=0$. Integrating once more, we find

$$u=\frac{p_2-p_1}{4\mu l}(r^2+C),$$

where C is to be determined from the boundary condition. In absence of sliding $u=0$ for $r=R$, whence $C=-R^2$, and therefore,

$$u=\frac{p_1-p_2}{4\mu l}(R^2-r^2).$$

The amount (volume) of liquid streaming through a cross-section is, per second, $2\pi\int_0^R urdr$. Thus, in our case,

$$\frac{\pi R^4}{8\mu l}(p_1-p_2), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is proportional to R^4, in agreement with Poiseuille's law.

In the presence of sliding the friction force $-\mu du/dr$ is equal and opposite to the force exerted upon the liquid by the tube walls. The boundary condition then becomes

$$-\mu\frac{du}{dr}=\lambda u, \text{ for } r=R,$$

whence

$$C=-R^2-\frac{2\mu}{\lambda}R,$$

and

$$u=\frac{p_1-p_2}{4\mu l}(R^2+\frac{2\mu}{\lambda}R-r^2).$$

The corresponding amount of liquid streaming across the section of the tube is, per second,

$$\frac{\pi R^4}{8\mu l}\left(1+\frac{4\mu}{\lambda R}\right)(p_1-p_2). \quad . \quad . \quad . \quad . \quad (8)$$

The term $4\mu/\lambda R$ gives the correction to Poiseuille's law for the sliding. For $\lambda=\infty$, or no sliding, (8) reduces to (7).

3. The Dragging of a Liquid by a Moving Plate

For a second illustration of the hydrodynamical equations let us take the case of a liquid contained between two flat plates of infinite extension, perpendicular to the Y-axis. Let the lower plate ($y=0$) be at rest, while the upper plate ($y=\Delta$) moves uniformly with the velocity a along the X-axis. Since the liquid is dragged by the upper plate, so that its velocity increases along the Y-axis, we will write for the velocity components

$$u=C_1+C_2y, \; v=0, \; w=0.$$

The friction upon any plane parallel to the plates is, per unit area,

$$\mu\frac{du}{dy}=\mu C_2.$$

For $y=0$ we have $u=C_1$, so that the relative velocity of the liquid and the lower plate is C_1, and therefore the force exerted by the latter upon the liquid λC_1. Thus the first boundary condition becomes

$$\mu\frac{du}{dy}=\lambda C_1.$$

This gives

$$\frac{\mu}{\lambda}C_2=C_1.$$

For $y=\Delta$, $u=C_1+C_2\Delta$, so that the relative velocity of the liquid and the upper plate is $a-(C_1+C_2\Delta)$, and the force exerted by this plate upon the liquid $\lambda(a-C_1-C_2\Delta)$. The friction of the liquid against this plate is $-\mu C_2$, and the second boundary condition thus becomes

$$\mu C_2=\lambda(a-C_1-C_2\Delta)$$

or, substituting the value of C_1,

$$C_2=\frac{\lambda a}{2\mu+\lambda\Delta},$$

whence the friction

$$\mu C_2=\frac{\mu a}{\frac{2\mu}{\lambda}+\Delta}.$$

For $\lambda=\infty$ we have $C_2=a/\Delta$, so that for finite λ the friction will be somewhat smaller.

Notice that μ/λ has the dimensions of a length. (This follows from the equation $\mu du/dy=\lambda C_1$, since C_1 is a velocity, to wit, $C_1=u$ for $y=0$.) The physical meaning of this length can be seen by imagining that either plate is moved away from the other over the distance μ/λ, while the liquid expands so as still to extend from plate to plate and its state of motion remains unchanged. The velocity of the liquid relatively to either plate is then nil at both boundaries. In fact, $u=0$ for $y=-\mu/\lambda$, and $u=a$ for $y=\Delta+\mu/\lambda$. Thus the solution of the problem with sliding can be reduced to that of the problem without sliding, provided the liquid is given the said expansion.

Also the result of the problem of a liquid flowing through a tube of circular section is in harmony with this property. For, if R in the expression (7) is replaced by $R+\mu/\lambda$, the expression (8) follows, provided μ/λ is small in comparison with R so that the square of $\mu/\lambda R$ can be neglected.

4. Effect of Sliding in the Case of Translational Motion of a Sphere in a Liquid

We will now determine the resistance offered by a liquid to a sphere endowed with uniform rectilinear motion. This will lead us to the famous *formula of Stokes* which, among other things, comes into play in the lately developed theory of Brownian movement. It is still an open question how far the validity of this formula can be upheld for very small particles and irregular motions. This will still be discussed in the sequel (Art. 7).

In dealing with the problem in hand we will assume that not the liquid as a whole is at rest and a sphere moves through it but, inverting the relations, we will imagine that the sphere is at rest and the liquid moves past it, having at infinity a uniform rectilinear motion. We put the Z-axis along the direction of this motion and take the centre of the sphere as the origin of our co-ordinate system. Thus, if u, v, w be the velocity components of the liquid, we have at infinity $u=0$, $v=0$, $w=a$.

From our previous equations (6)

$$\mu\Delta u=\frac{\partial p}{\partial x},\ \mu\Delta v=\frac{\partial p}{\partial y},\ \mu\Delta w=\frac{\partial p}{\partial z},$$

and from the equation of continuity (1) we derive

$$\Delta p=0.$$

To solve these equations we will follow Kirchhoff and introduce an auxiliary function Φ, such that

$$\mu\Delta\Phi=p.$$

To begin with, we could try as a solution

$$u=\frac{\partial\Phi}{\partial x},\ v=\frac{\partial\Phi}{\partial y},\ w=\frac{\partial\Phi}{\partial z}.$$

This satisfies the equations of motion, but not the equation of continuity. In fact, since

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\Delta\Phi,$$

the equation of continuity could only be satisfied if $\Delta\Phi=0$,

i.e. if the pressure were everywhere nil. We try, therefore, to help matters by adding new terms and write

$$u=\frac{\partial\Phi}{\partial x}+u',\ v=\frac{\partial\Phi}{\partial y}+v',\ w=\frac{\partial\Phi}{\partial z}+w'. \quad . \quad . \quad (9)$$

If the new terms can be so chosen that

$$\Delta u'=0,\ \Delta v'=0,\ \Delta w'=0 \quad . \quad . \quad . \quad (10)$$

and

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=-\Delta\Phi=-\frac{p}{\mu}, \quad . \quad . \quad . \quad (11)$$

our solution will be ready. Now, noticing that we must have $\Delta p=0$ and that this is satisfied by spherical harmonics of which the simplest are $1/r$ and all partial derivatives of $1/r$, we try to put

$$p=2c\mu\frac{\partial}{\partial z}\left(\frac{1}{r}\right).$$

Account is here taken of the circumstance that the pressure p must be an odd function of z, since it must have different signs at the opposite poles of the sphere. Further, the function of z chosen is the simplest which satisfies $\Delta p=0$. Finally, a constant factor is inserted which will presently be given a suitable value. An additive constant would for the problem in hand be without significance, and is therefore omitted.

Equations (10) and (11) are now satisfied by

$$u'=0,\ v'=0,\ w'=-\frac{2c}{r}.$$

Next, we put

$$\Phi=az+b\frac{\partial}{\partial z}\left(\frac{1}{r}\right)+c\frac{\partial r}{\partial z}.$$

The first two terms do not contribute to $\Delta\Phi$, and since

$$\Delta\left(c\frac{\partial r}{\partial z}\right)=c\frac{\partial}{\partial z}(\Delta r)=2c\frac{\partial}{\partial z}\left(\frac{1}{r}\right),$$

the last term gives

$$\Delta\Phi=\frac{p}{\mu}.$$

Notice that all three terms of Φ, though odd functions of z, are even functions of x and y, which harmonises with the symmetry of the liquid motion.

The first term gives at infinity, where the derivatives of the remaining two vanish,

$$w = \frac{\partial \Phi}{\partial z} = a.$$

Thus is a introduced into the formula. The middle term enables us to satisfy the boundary conditions at the surface of the sphere. By means of these conditions the constants b and c can be expressed in terms of the velocity a and the radius R.

Thus the solution becomes

$$\left.\begin{aligned} p &= 2\mu c \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \\ u &= \left(\frac{3b}{r^2} - c\right)\frac{xz}{r^3}, \quad v = \left(\frac{3b}{r^2} - c\right)\frac{yz}{r^3} \\ w &= a + \frac{b}{r^3}\left(\frac{3z^2}{r^2} - 1\right) - \frac{c}{r}\left(\frac{z^2}{r^2} + 1\right) \end{aligned}\right\}. \quad . \quad . \quad . \quad (12)$$

Thus far we have followed Kirchhoff. In the boundary conditions at the surface of the sphere we will take account of the sliding and in that deviate from Kirchhoff. Since at the surface of the sphere the velocity of the liquid must be tangential, the first boundary condition is

$$ux + vy + wz = 0, \text{ for } r = R,$$

whence

$$a + \frac{2b}{R^3} - \frac{2c}{R} = 0. \quad . \quad . \quad . \quad . \quad (13)$$

Equations (12) show that there is symmetry around the Z-axis. Let us then consider a point P at the surface of the sphere in the plane XZ, for which $\angle POZ = \theta$, and let us introduce a new orthogonal system of axes, of which OZ' passes through P and OX' lies in the plane XZ. The equations (12), when transformed to the new axes, become

$$\left.\begin{aligned} u' &= -\left(a - \frac{b}{r^3} - \frac{c}{r}\right)\sin\theta + \left(\frac{3b}{r^5} - \frac{c}{r^3}\right)x'(z'\cos\theta - x'\sin\theta) \\ v' &= \left(\frac{3b}{r^5} - \frac{c}{r^3}\right)y'(z'\cos\theta - x'\sin\theta) \\ w' &= \left(a - \frac{b}{r^3} - \frac{c}{r}\right)\cos\theta + \left(\frac{3b}{r^5} - \frac{c}{r^3}\right)z'(z'\cos\theta - x'\sin\theta) \end{aligned}\right\}. \quad (14)$$

The velocity of sliding at P follows from these equations for $x'=y'=0$, $z'=R$. Thus, and by (13), its components are

$$u'=-\left(a-\frac{b}{R^3}-\frac{c}{R}\right)\sin\theta,\ v'=0,\ w'=0.$$

Whence, the tangential tension at P exerted by the sphere upon the liquid,

$$\lambda\left(a-\frac{b}{R^3}-\frac{c}{R}\right)\sin\theta,$$

and the friction component, which must be in equilibrium with this force,

$$X'_{z'}=\mu\left(\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial x'}\right).$$

Thus the boundary condition becomes

$$\mu\left(\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial x'}\right)=-\lambda\left(a-\frac{b}{R^3}-\frac{c}{R}\right)\sin\theta,$$

where $z'=R$ and $x'=0$.

Evaluating the left-hand member by means of (14), one finds

$$X'_{z'}=-\frac{6\mu b}{R^4}\sin\theta,$$

and the second boundary condition assumes the form

$$\frac{6\mu b}{R^4}=\lambda\left(a-\frac{b}{R^3}-\frac{c}{R}\right). \quad . \quad . \quad . \quad . \quad (15)$$

Since θ has disappeared, this boundary condition can at once be satisfied all over the surface of the sphere.

Formulae (13) and (15) give

$$\frac{b}{R^3}=\frac{a}{4\left(1+\frac{3\mu}{\lambda R}\right)},\quad \frac{c}{R}=\tfrac{3}{4}a\frac{1+\frac{2\mu}{\lambda R}}{1+\frac{3\mu}{\lambda R}}.$$

Thus b and c are expressed in terms of known data, and substituting these values into (12), we find also u, v, w.

The total force exerted on the sphere by the liquid will fall, by reasons of symmetry, into the Z-axis and can thus be found by integrating $Z_{z'}$ over the surface of the sphere. Now, $Z_{z'}$ can be determined in two ways, either from

$$Z_{z'}=Z_x\sin\theta+Z_z\cos\theta,$$

or from the tangential force $X'_{z'}$ and the normal force $Z'_{z'}$ leading to

$$Z_{z'} = Z'_{z'} \cos\theta - X'_{z'} \sin\theta.$$

To choose the latter way, $Z'_{z'}$ must be evaluated from the equation

$$Z'_{z'} = -p + 2\mu \frac{\partial w'}{\partial z'} \text{ for } x' = 0,\ z' = R,$$

which gives

$$Z'_{z'} = \frac{6\mu c}{R^2} \cos\theta - \frac{12\mu b}{R^4} \cos\theta,$$

and since we have already found

$$X'_{z'} = -\frac{6\mu b}{R^4} \sin\theta,$$

we have

$$Z_{z'} = \frac{6\mu b}{R^4}(1 - 3\cos^2\theta) + \frac{6\mu c}{R^2} \cos^2\theta,$$

whence, by integration over the surface of the sphere, the required resistance,

$$W = 2\pi R^2 \int_0^\pi Z_{z'} \sin\theta d\theta = 8\pi c\mu,$$

i.e.

$$W = 6\pi\mu a R \frac{1 + \frac{2\mu}{\lambda R}}{1 + \frac{3\mu}{\lambda R}}.$$

For $\lambda = \infty$ this expression reduces to

$$W = 6\pi\mu a R,$$

which is Stokes' well-known formula.

Such, then, is the resistance experienced by a sphere moving with velocity a through a stagnant liquid.

5. Liquid Motion due to an Immersed Vibrating Plate

We shall next investigate how far this resistance formula can be assumed to hold for a sphere in non-uniform motion, such as the Brownian movement. The problem offers considerable difficulties, and we shall, therefore, confine ourselves

to the simple case in which the body has a vibratory motion of translation.

As an introduction we will consider a flat plate in the YZ-plane maintained in non-damped vibration in the direction of the Z-axis. Let the plate be unlimited, so that the state will be the same all along the Y- and the Z-axes and thus depend on x and t only. The velocities being again assumed to be infinitesimal, and all relevant magnitudes being functions of x and t only, the equation of motion

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\Delta w$$

reduces to

$$\rho\frac{\partial w}{\partial t}=\mu\frac{\partial^2 w}{\partial x^2}. \quad . \quad . \quad . \quad . \quad (16)$$

As its solution we take

$$w=ae^{int+\beta x}.$$

Substituting this into (16), we have

$$in\rho=\mu\beta^2, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

whence

$$\beta=-\sqrt{\frac{n\rho}{2\mu}}(1+i).$$

Thus,

$$w=ae^{int-\sqrt{\frac{n\rho}{2\mu}}(1+i)x},$$

of which the real part is

$$w=ae^{-\sqrt{\frac{n\rho}{2\mu}}x}\cos\left(nt-\sqrt{\frac{n\rho}{2\mu}}x\right),$$

representing waves which issue from the plate along the positive X-axis, this result being obtained by taking for β as solution of (17) the negative root. The wave-length is $2\pi\sqrt{2\mu/n\rho}$. That this has the dimensions of a length is manifest by (16). The amplitude of the oscillations of the liquid is thus decreasing considerably when these are propagated over a wave-length, namely, to $e^{-2\pi}$ times its original value.

6. Effect of Frequency in the Case of a Sphere vibrating in a Liquid

Passing to the case of a sphere vibrating in the direction of the Z-axis, we may represent its velocity component w by ae^{int}, where n is real, corresponding to non-damped vibrations. The motion of the liquid must satisfy the equations of motion

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\Delta u, \text{ etc.},$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

while at the surface of the sphere, if sliding be disregarded, $u = v = 0$, $w = ae^{int}$.

Of these equations two particular solutions can be found, whose superposition gives a solution which satisfies the boundary conditions at the surface of the sphere.

For the first solution we put $p = 0$. This will then exhibit some similarity with the problem of the vibrating plate, since in the case of the latter the equation of motion (16) was free of terms containing p.

Let us, therefore, introduce an auxiliary function Φ satisfying the equation

$$\rho\frac{\partial \Phi}{\partial t} = \mu\Delta\Phi$$

and depending on t and r only. The equations of motion will be satisfied not only by Φ itself but by its derivatives as well, and we can put

$$u = \frac{\partial^2\Phi}{\partial x\partial z}, \quad v = \frac{\partial^2\Phi}{\partial y\partial z}, \quad w = -\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right).$$

Of these the last is so chosen as to satisfy the equation of continuity.

The differential equation for Φ becomes, in polar co-ordinates,

$$\rho\frac{\partial(r\Phi)}{\partial t} = \mu\frac{\partial^2(r\Phi)}{\partial t^2},$$

this being of exactly the same form as the equation of motion

(16) of the liquid containing a vibrating plate. The solution is thus

$$\Phi = \frac{b}{r} e^{int+\beta r},$$

where $\beta = -\sqrt{\frac{n\rho}{2\mu}}(1+i)$ and b is a constant which for the present may be left undetermined. Thus far the first solution.

The peculiarity of the second solution is that it annihilates the last terms in the right-hand members of the equations of motion. As such we introduce a function χ of t and r which satisfies the equation

$$\Delta\chi = 0.$$

If then u, v, w are equalled to χ or to any of its derivatives, we shall have $\Delta u = \Delta v = \Delta w = 0$.

In order to satisfy the equations of motion and the equation of continuity as well we put

$$\rho u = \frac{\partial^2\chi}{\partial x \partial z},\ \rho v = \frac{\partial^2\chi}{\partial y \partial z},\ \rho w = \frac{\partial^2\chi}{\partial z^2},\ p = -\frac{\partial^2\chi}{\partial z \partial t},$$

and we take for χ the simplest harmonic function $\left(\frac{1}{r}\right)$ multiplied by the factor e^{int}, to express non-damped vibrations, and by the density ρ, to do justice to the equations of motion; in fine,

$$\chi = \frac{c\rho}{r} e^{int},$$

where the constant c is again left undetermined for the present.

Thus the general solution of our equations becomes

$$u = \frac{\partial^2\Phi}{\partial x \partial z} + \frac{1}{\rho}\frac{\partial^2\chi}{\partial x \partial z},$$

$$v = \frac{\partial^2\Phi}{\partial y \partial z} + \frac{1}{\rho}\frac{\partial^2\chi}{\partial y \partial z},$$

$$w = -\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \frac{1}{\rho}\frac{\partial^2\chi}{\partial z^2}.$$

The amplitudes of both functions can be compounded by putting

$$f = \frac{b}{r} e^{\beta r} + \frac{c}{r}.$$

Next, if we introduce the amplitudes u', v', w', defined by $u = u'e^{int}$, etc., these will satisfy the equations

$$u' = \frac{\partial^2 f}{\partial x \partial z},\ v' = \frac{\partial^2 f}{\partial y \partial z},\ w' = -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

The boundary conditions at the surface of the sphere are, in absence of sliding, $u' = 0$, $v' = 0$, $w' = a$, for $r = R$.

The first of these gives

$$\frac{df}{dr} - r\frac{d^2 f}{dr^2} = 0,$$

whence

$$(3 - 3\beta R + \beta^2 R^2) b e^{\beta R} + 3c = 0. \quad . \quad . \quad . \quad (18)$$

The second, $v' = 0$, gives the same equation, while the third boundary condition leads to

$$-\frac{2}{r}\frac{df}{dr} + \frac{x^2 + y^2}{r^3}\frac{df}{dr} - \frac{x^2 + y^2}{r^2}\frac{d^2 f}{dr^2} = a,$$

or, since $df/dr - rd^2 f/dr^2 = 0$,

$$-\frac{2}{r}\frac{df}{dr} = a, \text{ for } r = R,$$

whence

$$(1 - \beta R) b e^{\beta R} + c = \tfrac{1}{2} a R^3. \quad . \quad . \quad (19)$$

From (18) and (19) we have, for b and c,

$$b e^{\beta R} = -\frac{3}{2}\frac{a}{\beta^2} R, \quad . \quad . \quad . \quad . \quad (20)$$

$$c = (\tfrac{3}{2} - \tfrac{3}{2}\beta R + \tfrac{1}{2}\beta^2 R^2)\frac{aR}{\beta^2}. \quad . \quad . \quad . \quad (21)$$

Thus the state of motion is determined.

In view of the symmetry with respect to the Z-axis the resistance opposed by the liquid to the motion of the sphere can be represented by

$$2\pi R^2 \int_0^\pi Z_n \sin\theta d\theta,$$

where θ is the angle between the normal n (of the sphere) at a point of the XZ-plane and the Z-axis.

If we put $Z_n = Z'_n e^{int}$, $Z_x = Z'_x e^{int}$, etc., then

$$Z'_n = \frac{1}{R}(xZ'_x + zZ'_z),$$

$$Z'_x = \mu\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right),$$

$$Z'_z = -p' + 2\mu\frac{\partial w'}{\partial z}.$$

Evaluating Z'_n by means of the condition $df/dr - rd^2f/dr^2 = 0$, for $r = R$, the equations (20), (21), and the relation (17), we find for the required resistance, $K = 2\pi R^2 \int_0^\pi Z_n \sin\theta d\theta$,

$$K = 4\pi R(-\tfrac{3}{2} + \tfrac{3}{2}\beta R - \tfrac{1}{8}\beta^2 R^2)\mu a e^{int}. \quad . \quad . \quad (22)$$

For very slow vibrations, *i.e.* for very small values of βR, this reduces to Stokes' formula

$$K = -6\pi\mu R a e^{int} = -6\pi\mu R w_{sphere}.$$

For any frequency of vibrations (22) can be written

$$K = (g + ih)\mu a e^{int}.$$

The velocity of the sphere will then be

$$w = a\cos nt,$$

and the force opposed to the sphere by the liquid,

$$K = \mu g a\cos nt - \mu h a \sin nt. \quad . \quad . \quad . \quad (23)$$

Only the first term is to be considered as a resistance, if "resistance" be so characterised that the work done by it over a complete period is negative. Now, the work of K is

$$\int Kw dt = \int(\mu g a^2 \cos^2 nt - \mu h a^2 \sin nt \cos nt) dt,$$

and here the first term, whose coefficient g will presently appear to be negative, gives on integration over a complete period a negative quantity, whereas the second term gives nothing. The term $-\mu h a \sin nt$ is a force proportional to the acceleration; the corresponding effect is thus an *apparent increase of the mass* of the sphere, which is due to the co-vibration of the liquid.

Let us still consider the resistance $\mu g a \cos nt$. Here g is the real part of

$$4\pi R(-\tfrac{3}{2}+\tfrac{3}{2}\beta R-\tfrac{1}{6}\beta^2R^2),$$

and since

$$\beta=-(1+i)\sqrt{\frac{n\rho}{2\mu}},$$

we have

$$g=-6\pi R\left(1+R\sqrt{\frac{n\rho}{2\mu}}\right),$$

and the resistance becomes, with w_s written for the velocity of the sphere,

$$-6\pi R\mu w_s\left(1+R\sqrt{\frac{n\rho}{2\mu}}\right). \quad . \quad . \quad . \quad (24)$$

This is again the resistance according to Stokes, but increased by a term which for high frequencies can outweigh the Stokes resistance.

7. The Question of Validity of Stokes' Law for Brownian Movement

We can now ask what this resistance is like when the sphere is endowed with any variable motion. This is of importance in connection with Brownian movement. But we must limit ourselves to small velocities, so as to be able to neglect, as above, all terms such as $u\partial u/\partial x$, etc. If the velocity were known as a function of the time for the whole duration of the experiment, it could be developed into a Fourier series, and the result (24) could be applied to each term. Thus the resistance would be found. This, however, is not very helpful for a general discussion.

Stokes' formula cannot be applied to the case of Brownian movement, as this is much too quickly variable for such a purpose. The term $R\sqrt{\frac{n\rho}{2\mu}}$ in (24) is to be neglected in presence of unity, if the vibration time T is large compared with $\pi\rho R^2/\mu$. If we write $\pi\rho R^2/\mu=\theta$, then it will be possible to apply Stokes' law for slow vibrations, for which, that is, the vibration time T is large in comparison with θ, and this holds also for other

motions, provided the velocity does not change much during the time θ.

To illustrate this by an example, let us see whether Stokes' law is applicable to the extinction of the motion of a sphere due to the friction of the liquid. If m be the mass and v the velocity of the sphere, Stokes' law would give

$$m\frac{dv}{dt} = -6\pi\mu R v,$$

whence

$$v = v_0 e^{-\frac{6\pi\mu R}{m}t},$$

so that v would dwindle down to the e-th part of v_0 after the time $m/6\pi\mu R$. If ρ_1 be the density of the sphere, this time is

$$\tau = \frac{m}{6\pi\mu R} = \frac{2}{9}\frac{\rho_1 R^2}{\mu}.$$

If ρ and ρ_1 are comparable with each other, θ is seen to be of the same order as the time τ. Stokes' law cannot, therefore, be applied to the extinction of motion here considered.

For particles in Brownian movement θ becomes quite small. If $R = 5 \cdot 10^{-5}$ and $\mu = 18 \cdot 10^{-5}$, then

$$\theta = \frac{\pi \cdot 25 \cdot 10^{-10}}{18 \cdot 10^{-5}}\rho = 4 \cdot 10^{-5}\rho,$$

approximately. Yet the motion of the suspended particles will vary during this time considerably, so that the refinements of the Brownian movement cannot be mastered by the law of Stokes. In many other cases, however, the motion within the time θ will change but little and the law will be applicable.

8. Deduction of Einstein's Formula for the Mean Square of the Deviation of a Particle in Brownian Movement

For the investigation of Brownian movement it is of importance to correlate the mean square ξ^2 of the distance attained by a particle within a given time t with the properties of the liquid, *i.e.* with the viscosity coefficient μ, and with the radius R of the particle.

Without following the particle's actual crinkly path and its rapidly variable motion, we can find the required connection by

a roundabout way, namely, by considering the diffusion velocity of the particles. On the one hand this can be expressed by μ, and on the other hand by $\overline{\xi}^2$. Evaluating thus the diffusion coefficient by two different methods and equating the two expressions, we shall find Einstein's formula * for $\overline{\xi}^2$.

First method. Suppose we had a liquid containing suspended particles whose concentration varies in some direction or other. Along that direction put the X-axis. The concentration of the particles is given by their number n per unit volume. Thus n is a function of x. The particles will exert an osmotic pressure. Now, if it be assumed that the mean kinetic energy of a particle is equal to that of a gas molecule at the same temperature T, *i.e.* $\frac{3}{2}kT$, the osmotic pressure is

$$p = nkT,$$

this being $\frac{2}{3}$ of the total kinetic energy of the particles per unit volume.

The force driving the particles is equal to the difference of their osmotic pressure in two planes perpendicular to the x-axis. In a stationary state of diffusion this difference will be balanced by the force exerted by the liquid upon the particles.†

Apart from the Brownian movement the particles have a common velocity along the X-axis if the concentration decreases with increasing x.

Since the formulae for the stress components and the equations of motion of the liquid are linear (products of velocities and derivatives of velocities being omitted, as they are small), we may say that the force exerted by the liquid upon the particles consists of two parts, one corresponding to their common velocity, and another to their Brownian movement. The latter will vanish for all the particles taken together.

Thus we have

$$-\frac{dp}{dx} = 6\pi\mu Rnv,$$

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* *Ann. der Physik*, vol. xix., 1906, p. 371.

† The osmotic pressure is, properly speaking, the momentum along the X-axis transferred by the moving particles in the positive less than in the negative direction, per unit area perpendicular to this axis and per unit time. Consequently, the momentum of particles contained in a layer of thickness dx undergoes the change $-\frac{dp}{dx}dx$ per unit time, *i.e.* as if a force of this magnitude acted upon the particles.

where the left-hand member represents the gradient of the osmotic pressure, and the right-hand member the resistance opposed by the liquid to n particles; and since $p = nkT$,

$$-kT\frac{dn}{dx} = 6\pi\mu Rnv.$$

Whence the diffusion current

$$nv = -\kappa\frac{dn}{dx},$$

where κ, the coefficient of diffusion, has the value

$$\kappa = \frac{kT}{6\pi\mu R}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The diffusion is thus calculated by means of the law of Stokes.

In the second method we concentrate all our attention upon the Brownian movement. We do not consider, however, the actual path of a particle, but its total displacement within a certain time. Let the distance attained after a time t, reckoned from some initial moment, be s. Then s is different for different particles. By means of probability considerations it can be shown that the mean value of s^2 is, for all particles, proportional to t, say,

$$\overline{s^2} = \beta t.$$

The coefficient β can be observed, as, among others, was done by Perrin.*

Thereupon will our diffusion theory be based. Suppose that the concentration varies from point to point; then the particles from a small volume-element will spread after a time t over a sphere of radius $\sqrt{\beta t}$. Thus the concentration differences will be gradually obliterated.

Now, it can be shown that

$$\kappa = \frac{1}{2}\frac{\overline{\xi^2}}{t}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where ξ is the projection of s upon the X-axis.†

* *Comptes rendus*, Paris, vol. cxlix., 1909, p. 477, and vol. clii., 1911, p. 1569.

† Suppose that during the time t all particles are displaced along the X-axis over the same distance l, so that for one half of the particles x is increased, and for the remaining half diminished by l. Next, consider a plane V, perpendicular to the X-axis, and two layers, each of thickness l, on both sides of V. Let N_1 be the number of particles contained, at the beginning of the time t

From (25) and (26) follows Einstein's formula

$$\frac{\overline{\xi^2}}{t} = \frac{kT}{3\pi\mu R} = \frac{RT}{N}\frac{1}{3\pi\mu R}, \quad . \quad . \quad . \quad (27)$$

where R is the gas constant and N the number of molecules per gram molecule.

By means of this formula Perrin determined N from his experiments.

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and per unit area of V, in the layer on the positive side, and similarly N_2 on the negative side of V. Then the diffusion per unit time is $\frac{1}{2}(N_2 - N_1)/t$. We can assume that within these thin layers n is a linear function of x, and shall thus find, for the diffusion, $-\frac{1}{2}\frac{dn}{dx}\frac{\overline{l^2}}{t}$, and, for the coefficient of diffusion,

$$\kappa = \frac{1}{2}\frac{\overline{l^2}}{t},$$

which obviously must be replaced by (26), when account is taken of the diversity of the ξ-values for different particles.

It will be readily seen that $\overline{\xi^2} = \frac{1}{3}\overline{s^2}$.

CHAPTER II

FRICTION AND SLIDING, TREATED KINETICALLY

9. Friction Independent of the Density of the Gas

Returning to the questions concerning inner friction and sliding, we will now treat them on the kinetic theory. Moreover, instead of a liquid, we shall now consider a gas. If this be strongly condensed, there is no sliding. Thus we can at first exclude the sliding, to introduce it later on as a correction.

We will begin by proving that the inner friction is independent of the gas density. For this purpose we consider a simple case of motion, viz. a gas whose horizontal layers are, as a whole, shifted over each other. Let us introduce a co-ordinate system whose XZ-plane is parallel to these layers, the X-axis pointing in the direction of streaming. If u be the velocity of a layer, the motion of the gas can be expressed by the equation

$$u = cy,$$

where c is a constant, it being assumed that the layer $y = 0$ is at rest.

The state of a molecule will be determined by ξ, η, ζ, the velocity components, and x, y, z, the co-ordinates of its centre of gravity. The relative co-ordinates and velocities of the parts of a molecule with respect to its centre of gravity can here be disregarded, as they do not affect our problem. Let us consider the molecules which at the instant t are contained in the volume-element $dS = dxdydz$ at the point P, of co-ordinates x, y, z, and whose velocity components are contained between ξ and $\xi + d\xi$, η and $\eta + d\eta$, ζ and $\zeta + d\zeta$. The number of these molecules can be expressed by

$$F(\xi, \eta, \zeta, y, t)dSd\lambda,$$

where $d\lambda = d\xi d\eta d\zeta$ and F is independent of x and z, since, by assumption, the state is the same throughout a gas layer parallel to the XZ-plane.

Further, let $bdSd\lambda dt$ be the number of molecules which are thrown into this group by collisions during the time dt, and $adSd\lambda dt$ the number of those which for the same reason and during the same time leave this group. Let external forces be absent. At the time $t+dt$ the molecules of this group will come to lie in a volume-element $dS' = dS$ [by Liouville's theorem], at the point $x+\xi dt$, $y+\eta dt$, $z+\zeta dt$. Thus,

$$F(\xi, \eta, \zeta, y, t)dSd\lambda + (b-a)dSd\lambda dt$$

will be the number of molecules contained at the instant $t+dt$ in the element dS, constructed at the point $x+\xi dt$, $y+\eta dt$, $z+\zeta dt$, and having their velocity components within the domain $d\lambda$. Whence,

$$b - a = \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial t}.$$

This equation would enable us in general to find the properties of the function F. This, however, can be accomplished only in the simplest case. If we put $F = f$ for $y = 0$, then

$$F(\xi, \eta, \zeta, y, t)_{y=0} = f(\xi, \eta, \zeta, t).$$

If the whole gas is given a translation velocity $-cy$ along the x-axis, the state of motion in a plane at the height y becomes what it was originally in the plane $y = 0$. Whence we see that, generally,

$$F(\xi, \eta, \zeta, y, t) = f(\xi - cy, \eta, \zeta, t).$$

We assume further the state to be stationary, so that

$$\frac{\partial F}{\partial t} = 0, \quad b - a = \frac{\partial F}{\partial y}\eta.$$

It remains only to determine the state for $y = 0$. From the preceding equation we have, for $y = 0$,

$$b - a = -c\frac{\partial f}{\partial \xi}\eta. \quad . \quad . \quad . \quad . \quad (28)$$

We will now compare the state of the gas in the said motion with the state in which there is no streaming, *i.e.* for $c = 0$. For this case f can be written Nf_0 (where N is the number of molecules

per unit volume). Then, in the state of motion, $f = Nf_0 + f'$, where the function f' contains all the refinement; it may assume positive values for some groups, and negative for others. The exact determination of f' is laborious and for our purposes unnecessary, since it can be shown by a simple reasoning that the friction is independent of the density. We assume that the velocity gradient c is small, in other words, that there is only a small departure from the state of rest, so that f' is small compared with Nf_0. Then f in the right-hand member of (28) can be replaced by Nf_0, which amounts to neglecting only terms of a higher order. This gives

$$b - a = -cN \frac{\partial f_0}{\partial \xi} \eta.$$

For the state of rest Maxwell's velocity distribution * holds, and $b - a = 0$. Owing to the change brought about in the function f by the motion $b - a$ does no longer vanish. We consider again a certain group of molecules and we distribute also the remaining molecules into groups according to their velocities and co-ordinates. Let the numbers of molecules in these groups be n_1, n_2, etc. Owing to the collision of a molecule of the first group with a molecule of one of the remaining groups that molecule leaves its group. Such a collision then belongs to the type a. Collisions of molecules of the remaining groups with each other can increase the numerosity of the first group and will thus be of type b. Since the number of collisions between two groups of molecules is proportional to the numerosity of the first as well as to that of the second, we can write

$$b - a = \Sigma a_{12} n_1 n_2,$$

where a_{12} is positive for such collisions as contribute something to b, and negative for such as contribute to a.

For $c = 0$ the numbers of molecules in these groups will be denoted by n_{10}, n_{20}, etc., so that $b - a = \Sigma a_{12} n_{10} n_{20} = 0$.

In general we can put

$$b - a = \Sigma a_{12} (n_{10} + n_1')(n_{20} + n_2'),$$

---

* Viz.:

$$f_0(\xi, \eta, \zeta) = \sqrt{\frac{h^3}{\pi^3}} e^{-h(\xi^2 + \eta^2 + \zeta^2)},$$

where h is inversely proportional to the absolute temperature.

where the a's remain as before, since we are considering groups which are completely determined by their velocities, so that when we pass from the state of rest of the gas as a whole to its streaming only the numerosity of each of the groups is changed.

Since we consider only a small departure from the state of rest, n_1', n_2', etc., are small compared with n_{10}, n_{20}, etc., and therefore

$$b - a = \Sigma a_{12}(n_{10}n_2' + n_{20}n_1').$$

This leads to the equation

$$\Sigma a_{12}(n_{10}n_2' + n_{20}n_1') = -cN\frac{\partial f_0}{\partial \xi}\eta, \quad . \quad . \quad (29)$$

by means of which the variation of f or the numbers n_1', n_2', etc., can be determined. If the molecules are divided into K groups, there are K equations such as (29).

If we change c, the equation (29) remains satisfied, provided n_1', n_2', etc., are changed in the same ratio; thus the whole disturbance, and therefore also f', will be proportional to the velocity gradient. Now, if the density be varied, and thus also N, the number of molecules in each group for the state of rest, n_{10}, n_{20}, etc., varies in the same ratio. Then (29) is satisfied, provided n_1', n_2', etc., remain the same, so that also f' does not change.

Whence we see that f' is proportional to c and independent of N, that is to say, independent of the density.

The friction upon a plane parallel to the XZ-plane is, per unit area,

$$m\int \xi\eta f d\lambda.$$

In fact, this is the momentum along the X-axis, transferred per unit time across a unit area parallel to the XZ-plane. Since in the state of rest there is no friction,

$$m\int \xi\eta f_0 d\lambda = 0,$$

and the last expression for the friction can also be written

$$m\int \xi\eta f' d\lambda.$$

This, as f' itself, is proportional to c and independent of N, and can, therefore, be written

$$\mu c,$$

where μ is the coefficient of friction. The latter is thus seen to be independent of density.

10. Uniformity Considerations

That friction is independent of density can also be proved by another method which, though not going so deep into the phenomena, is yet exact. We shall in this case also confine ourselves to a simple example, but taking at the same time account of the sliding. We consider two infinitely extended horizontal plates, of which the lower is at rest and the upper moves with a constant velocity v horizontally, towards the right, and we propose to find the motion of the gas contained between the plates.

Fig. 1 gives a graphical representation of the motion of the gas for the case in which there is no sliding along the walls. The abscissae are the velocities in the different layers, and the ordinates the distances from the lower plate. The velocity v of the upper plate is represented by AB. Manifestly, the line joining the end-points of the velocity vectors is a straight line.

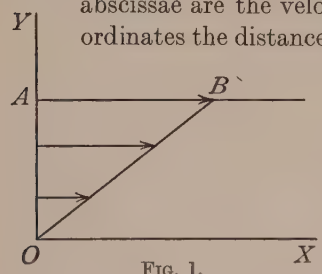

Fig. 1.

A sliding along the walls will have the effect that the velocity of the gas layer in contact with the upper plate will be smaller. If this be represented by AF (Fig. 2), then $AF<AB$. Similarly, the velocity of the layer touching the lower plate will no longer be nil, but will have some small value OE. The distribution of velocities over the different layers will no longer be represented by a straight line but by the line EF in Fig. 2, which, when the plates are far enough apart, has a straight portion, but is curved at the extremities.

To enter somewhat deeper into the sliding along the walls, we must keep in mind that by the velocity of a gas layer is meant the average velocity of the particles of which the layer at a given instant consists. Among the particles of the lowermost layer there will be some which previously belonged to a layer with some velocity of streaming and were carried down to that layer by collisions. For these a certain direction of motion, viz. that of the streaming velocity, will be privileged. In this layer, however, there will also be present, at the given instant, some particles which were there before and which collided with

the solid wall. We have, therefore, to distinguish also different cases of collision with the wall. If the wall is a perfect reflector, then the particles retain after the collision their tangential velocity component. But in general even the best polished walls will have to be considered as rough with regard to impacts of molecules, so that the reflection will be of a diffuse nature, and there will be no privileged direction of motion for the rebounding particles. But, as we just saw, there is such a direction for the impinging particles, so that, all things being considered, the layer in contact with the wall will have a velocity in the direction of streaming, very much as in the case in which there is sliding. The precise form of the curve representing the velocity distribution is hard to deduce from these considerations, at least for points near the plate, where the state of motion is very complicated. Among other things one would have to take into account that some of the molecules colliding with the wall adhere to it. But the farther away from the wall, the less the irregularities, and if we assume the mutual distance of the plates large compared with the mean free path of the molecules, a considerable portion of the curve will be straight. If the distance were very small, the straight-line portion would entirely disappear.

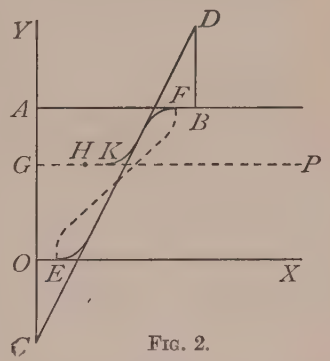

Fig. 2.

In Fig. 2 the velocity curve is represented by EF. (Probably its shape should be somewhat different, such, *e.g.*, as indicated by the dotted line.) If the straight-line portion of the curve be produced up to the points C and D, the former being on the Y-axis and the latter having the abscissa $AB = v$, then C and D will have the following significance. If the gas were replaced by an imaginary gas layer extending also beyond the plates and having throughout the same constant velocity gradient as that prevailing in the actual case only between the plates, then the layers of gas at C and D would have the same velocities as the plates, *i.e.* zero at C and v at D. (Cf. Art. 3, p. 82.)

We will now prove that the segments OC and BD are

independent of the distance of the plates. For this purpose let us imagine a plate P inserted at the height of G and moved in such a way that the state between G and A is not changed, except, of course, in the immediate neighbourhood of the plate P itself. We make the arc of the velocity curve near P congruent to that at E. Now if $HK = OE$, the velocity distribution represented by KF can prevail, provided GH is equal to the velocity of the plate P. In fact, if the whole system is given a common velocity equal and opposite to GH, the velocity curve retains the same shape, but is simply shifted over the distance GH to the left. The state at the plate P will then be exactly the same as that actually prevailing at the lowermost wall. Whence it follows that OC and BD are independent of the mutual distance of the plates.

In much the same way it can be shown that $OC = BD$. In fact, if the whole system is given a velocity equal and opposite to AB, the upper plate is brought to rest, while the lower one will have a velocity v towards the left, and the velocity curve will be shifted over a distance AB in the same direction. Thus, if the figure is turned around by 180°, it must be exactly the same as that representing the actual state. The shape of the curve at E must, therefore, coincide with that at F, and thus $OE = FB$ and also $OC = BD$.

Further, if the velocities of the plates are supposed to be very small compared with those of the molecules, the state of the gas can be considered as an infinitesimal deviation from the state of apparent rest. This deviation can then be put proportional to the infinitesimal cause, *i.e.* to the velocity v of the upper plate. All horizontal lines of the figure can thus be magnified or reduced in the same ratio as v without changing OC and BD. Whence we see that OC and BD are independent of the velocity v. Again, as we saw before, OC and BD remain also unaffected by the change of the distance between the plates. They can thus depend only on the nature and the density of the gas. All this, of course, holds only if the velocity curve has a rectilinear portion, and ceases, therefore, to be true when the distance between the plates is of the order of the mean free path of the molecules. If we put $OC = BD = \nu$, then ν will be characteristic for the gas. The meaning of ν is, that if the velocity gradient were constant not only between but also at the walls, the gas at a distance ν from the plate would attain the velocity of the plate itself.

Let the velocities of the plates be *zero* and v, and their mutual distance Δ. Then, by what precedes, the velocity gradient of the gas (*i.e.* the change of velocity per unit length in the direction of the Y-axis) will be $\frac{v}{\Delta + 2\nu}$, and, therefore, the friction per unit area, parallel to the plates, $\frac{\mu v}{\Delta + 2\nu}$.

We will call ν the coefficient of sliding and we will prove that it is inversely proportional to the density ρ, while it will be shown once more that μ is independent of ρ.

For this purpose let us consider a second system which will be denoted by II., while the original system will be denoted by I. ; the magnitudes relating to the system II. will be distinguished by dashes. Let the system II. be so chosen that the corresponding vertical distances are k times smaller than those in I., while the velocities at corresponding points are the same. We will now prove that the system II. represents a possible state of motion, provided the density in II. is k times that in I.

Let the function determining the state at a point P of the first gas be $F_1(\xi, \eta, \zeta)$, and that at a corresponding point P' of the second gas $F_2(\xi, \eta, \zeta)$. Then the condition for the density will imply that

$$F_2(\xi, \eta, \zeta) = kF_1(\xi, \eta, \zeta).$$

Since this holds for each group of molecules, the velocity of streaming, *i.e.* the mean value of ξ, will, as we assumed, be the same at P' as at P.

We have now to ascertain whether the condition of a possible state of motion is satisfied, and for this purpose we consider the equation for a stationary state of the gas, viz.

$$b - a = \frac{\partial F}{\partial x}\xi + \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial z}\zeta.$$

If the X-axis is again drawn towards the right, and the Y-axis perpendicularly to the plates, this equation becomes

$$b - a = \frac{\partial F}{\partial y}\eta.$$

It holds for the state I. and, as is easily seen, also for the state II. In fact, for the state II. both sides of the equation become k^2 times greater. For the left-hand member expresses

the number of collisions (and here the gas may be assumed homogeneous around P'), and if the number of molecules in each group is k times as large, the number of collisions is increased k^2 times. And with regard to the right-hand member, notice that η is the same for both systems, and that $\partial F_2/\partial y = k^2 \partial F_1/\partial y$, since $F_2 = kF_1$ and since all dimensions along the Y-axis in the system II. are k times smaller than those in the system I. Thus the equation of the stationary state within the gas is satisfied.

It remains, however, to consider the state at the walls. The number of molecules of a given group which, per unit time, strike against a wall, will for the system I. be determined by $\eta F_1 d\lambda$, where F_1 indicates the state at the considered point of the boundary layer. Similarly, this number for the system II. will be given by $\eta F_2 d\lambda$. Suppose, further, that the two walls are perfectly equal; then the collisions of molecules of the group considered will, in the system II., be k times more numerous than, but otherwise entirely the same as, those in the system I. Also for the molecules rebounding from the wall will the numbers, for groups with the same values of ξ, η, ζ, in the system II. be k times those in the system I. This, moreover, will be the case for perfectly reflecting as well as for diffusedly reflecting walls, the latter in contrast with the former being such that the molecules after reflection retain no trace of the mean motion relative to the wall which they had before the collision. If the above is assumed to hold also for the case that there is an adhering gas layer at the walls, then the system II. will represent also at the walls a possible state of motion.

Since the vertical dimensions in II. are k times smaller than those in the system I., we have (Fig. 2) $B'D' = \frac{1}{k} BD$ or

$$\nu' = \frac{\nu}{k}.$$

Thus it is proved that the sliding is inversely proportional to the density.

The friction, per unit area, upon a plane parallel to the XZ-plane will, as we saw before (p. 101), be represented by

$$w = \int m \xi \eta F d\lambda,$$

whence

$$w' = kw.$$

Since the velocity gradient in the system II. is k times that in I., and since $w = \mu$ times this gradient, we have

$$\mu' = \mu,$$

which proves that the friction is independent of the density.

11. Kundt and Warburg's Experimental Investigations

In 1875 an experimental investigation on friction and heat conduction in rarefied gases was published by Kundt and Warburg,* confirming the preceding theoretical results.

A round horizontal disc S, in bifilar suspension, was made to oscillate in a gas between two fixed plates S_1 and S_2. The coefficient of viscosity was determined by measuring the logarithmic decrement of the oscillations. This method was already applied by Coulomb in the case of a disc oscillating in an unlimited gas mass, while Maxwell improved it by introducing the two fixed plates, which has enabled him to make the calculation somewhat more accurate.

Kundt and Warburg's formula is

$$MD\beta = A\mu(1 + \theta),$$

where M is the moment of inertia of the vibrating disc, D the distance between the disc and one of the fixed plates, β the damping (appearing in the angular deviation $w = ae^{-\beta t} \cos nt$), and μ the viscosity coefficient, while θ is a number which depends on the density of the gas and the distance of the plates and which was accurately calculated by Maxwell. The spinning disc generates transversal friction waves. If the layer is thin enough, the air current is everywhere in phase with the disc, and $\theta = 0$. Such was actually the case in Kundt and Warburg's experiments. Again, A would be equal to $\frac{1}{2}\pi R^4$ for each plate surface (of radius R) exposed to the friction of the gas, provided that the friction experienced by a surface-element of the disc having a velocity v is taken as equal to the friction upon a surface-element of an infinitely extended plate moving with the translation velocity v between two fixed unlimited plates placed at the same distance. The value of A was more accurately calculated by Maxwell.†

* *Ann. Phys. und Chemie*, clv., 1875, pp. 337 and 525.
† *Phil. Trans.*, London, clvi., 1866, p. 249.

In these experiments the pressure p of the gas and the logarithmic decrement λ of the oscillations were measured. The latter decreases at smaller pressures, this being due to the influence of sliding. In fact, if for a large gas density, for which we can put $\nu=0$, the logarithmic decrement is λ_0, we have for λ, which is proportional to the friction, for smaller densities,

$$\frac{\lambda}{\lambda_0}=\frac{D^*}{D+2\nu}=\left(1+\frac{2\nu}{D}\right)^{-1}=\left(1+\frac{a}{p}\right)^{-1},$$

since

$$\frac{a}{p}=\frac{2\nu}{D}.$$

The factor a will, for a given distance of the plates, be a constant. Its value will vary inversely with this distance.

Here are the results of some of the experiments in which the gas was air, and $D=0{\cdot}1104$ cm. :

p	λ
320 mm.	0·1318
20	0·1306
7·7	0·1292
7·6	0·1292
2·4	0·1256
0·63	0·1109

For a large number of observations on various gases and at different pressures, Kundt and Warburg actually succeeded in choosing a so that the values of λ calculated by means of a, p, and λ_0 agreed with the observed ones. Some of these results were :

p	λ obs.	λ calc.
20 mm.	0·131	0·131
7·6	0·129	0·129
2·4	0·125	0·124
1·53	0·120	0·120

For different values of D the value of a appeared to be with fair accuracy inversely proportional to D. Thus for three different distances the values of a, when derived from the observations, were :

$$a_1=0{\cdot}149,\ a_2=0{\cdot}070,\ a_3=0{\cdot}061.$$

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* Cf. p. 105.

On the other hand, starting from a_1 and assuming that a_2 and a_3 are equal to $a_1 D_1/D_2$ and $a_1 D_1/D_3$, the result was :

$$a_2 = 0{\cdot}084,\ a_3 = 0{\cdot}059.$$

For still smaller pressures we are reduced to the case in which the straight portion of the velocity-curve (Fig. 2) is absent. According to Kundt and Warburg, the formula for λ continues to hold as long as the distance of the plates is at least 14 times the mean free path of the molecules.

Lastly, they find for air at the pressure of 76 cm. and the temperature of 15° C.

$$\nu = 0{\cdot}00001 \text{ cm.},$$

which shows that the sliding at normal pressure is small. But at the pressure of 1 mm. one should then still find $\nu = 0{\cdot}0076$ cm., so that for a distance $D = 0{\cdot}2$ cm. of the plates the effect of sliding would already be very marked. Indeed, the term 2ν in the expression $\dfrac{\mu v}{D + 2\nu}$ could not be neglected in the presence of D.

The mean free path of the molecules at normal pressure is about 0·0000084 cm., and thus of the same order as ν.

CHAPTER III

KNUDSEN'S INVESTIGATIONS ON RAREFIED GASES

12. Flow of a Rarefied Gas through a Narrow Tube

While in what precedes we have considered the extreme case in which the gas density is large, we will now turn to Knudsen's investigations * which represent the other extreme case, corresponding, that is, to such a small density that the effect of collisions can be disregarded. Knudsen investigates, theoretically as well as experimentally, the flow of such a strongly rarefied gas through a cylindrical tube whose walls are supposed to be so rough that the rebounding molecules have no trace left of their original streaming motion, being reflected from the walls in all possible directions.

Obviously the problem in hand can be treated on similar lines to that of heat radiation. Analogously to the latter we will write

$$A \cos \theta d\omega d\sigma$$

for the number of particles rebounding from an element $d\sigma$ of the wall within a cone of aperture $d\omega$, the axis of the cone being inclined at an angle θ to the surface normal. This amounts to assuming that, with respect to the reflected particles, the normal of the surface-element of the wall is privileged. The factor A depends on the total number of particles rebounding from the element $d\sigma$ of the wall, and this is, for a stationary state, equal to the number of particles striking this wall-element. Thus A is a function of the state of the gas.

* *Ann. der Physik*, xxviii., 1909, p. 75.

Similarly as for heat radiation, the number of particles which two elements $d\sigma$ and $d\sigma'$ send to each other can be represented by

$$A \cos\theta d\omega d\sigma = A \cos\theta' d\omega' d\sigma' = \frac{1}{r^2} A \cos\theta \cos\theta' d\sigma d\sigma',$$

where $r = PP'$ (Fig. 3).

Thus, in a space occupied by a strongly rarefied gas a state of stationary equilibrium, analogous to that of black body radiation, is possible, provided the walls are kept at a constant temperature. For all elements of the walls the value of A is the same, and the last expression holds also for the number of particles which pass through two surface-elements placed within the gas.

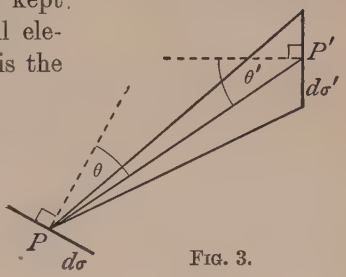

FIG. 3.

We will assume for the gas Maxwell's law of velocity distribution, and consider, in the first place, the number of particles which traverse, per unit time, a surface-element $d\sigma$ at a point P within the gas, and whose direction of motion falls within a cone of angle $d\omega$ and axis coinciding with the normal of the surface-element. This number can be written

$$A d\sigma d\omega,$$

and, if $f(v)dv$ be the number of particles per unit volume whose velocity falls within the interval v to $v+dv$,

$$\int_0^\infty v d\sigma \frac{d\omega}{4\pi} f(v)dv = A d\sigma d\omega.$$

Thus,

$$A = \frac{1}{4\pi}\int_0^\infty v f(v)dv.$$

According to Maxwell's law,

$$f(v) = Ce^{-hv^2}v^2,$$

where h is inversely proportional to the temperature and C depends on the density of the gas. This gives

$$A = \frac{C}{8\pi h^2}.$$

Since the gas pressure is equal to two-thirds of the kinetic energy per unit volume, we have

$$p = \tfrac{2}{3} \cdot \tfrac{1}{2} mC \int_0^\infty e^{-hv^2} v^4 dv = \tfrac{1}{8} mC \sqrt{\frac{\pi}{h^5}},$$

whence

$$A = \frac{p}{m\pi} \sqrt{\frac{h}{\pi}}.$$

Thus A is a function of the temperature [through h] and of the pressure.

At constant pressure, A diminishes with increasing temperature. This can be seen directly. In fact, at a higher temperature the molecules strike more vigorously; if, therefore, the pressure is still to remain the same, there must be a smaller number of collisions.

Let us now consider a gas flowing in a vertical cylindrical tube downward along the axis, as in Fig. 4, where the X-axis is along the axis of the tube.

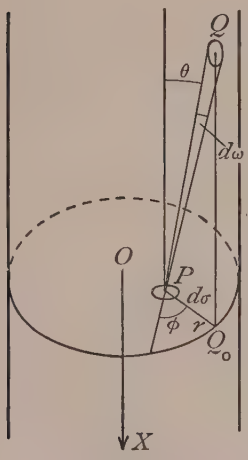

FIG. 4.

We will assume the temperature to be throughout the same. If such were also the case of the pressure, the gas would remain in the state of equilibrium described above, and A would be throughout the same. But in our case the pressure increases up the tube, and the same will also be true of A. We suppose p to be slowly variable, so that it can be represented by a linear function of x, which amounts to retaining of the Taylor-series development of p the first term only. In accordance with this assumption of small pressure variations, the values of A will be the same in all points of a cross-section of the tube. In fact, every surface-element can be assumed to be hit by as many molecules as if the pressure were everywhere the same as in the neighbourhood of that element. For the number of particles arriving from above is as much increased as that of particles moving upwards is diminished, so that the

total number crossing the surface-element is not changed. A is thus a function of x alone, and this holds for every tube, no matter what the shape of its cross-section.

Let us now consider an element $d\sigma$ of the section $x=0$ placed at the point P (Fig. 4), and let us find the number of particles crossing this element per unit time in a bundle of directions contained within the cone of angular aperture $d\omega$ whose axis makes an angle θ with the normal. This cone cuts the wall in an element placed at a point Q. Thus, the particles in question issue from an element $d\sigma'$ of the wall within the cone aperture $d\omega'$, the axis of this cone making an angle θ' with the normal, so that their number can be written

$$A_Q \cos\theta' d\omega' d\sigma'.$$

By what was said before this can be replaced by

$$A_Q \cos\theta d\omega d\sigma.$$

Here A_Q is the value of A at the point Q. Let Q_0 be the intersection point of the generatrix of the cylinder [passing through Q] with the contemplated section, and let $PQ_0=r$. Then

$$QQ_0 = r\cot\theta.$$

Again,

$$A_Q = A - r\cot\theta\frac{dA}{dx},$$

where A and dA/dx are the values of these magnitudes at the section $x=0$.

Thus the required number of molecules will be

$$A\cos\theta d\omega d\sigma - \frac{dA}{dx} r\cos\theta\cot\theta d\omega d\sigma.$$

Integrating this expression with respect to $d\omega$ over a whole sphere, we shall find the current through the element $d\sigma$. Now, the first part contributes nothing to this integral, while the second gives

$$-2\frac{dA}{dx} d\sigma \int r\cos\theta\cot\theta d\omega,$$

where the integral is to be extended over the upper half of the sphere.

Thus we can write for the current across the whole section

$$-G\frac{dA}{dx},$$

where $$G=2\int d\sigma\int r\cos\theta\cot\theta d\omega.$$

Now, if ϕ be the angle contained between r and a fixed direction in the section,

$$d\omega=\sin\theta d\theta d\phi,$$

and

$$G=2\int d\sigma\int_0^{2\pi}rd\phi\int_0^{\pi/2}\cos^2\theta d\theta=\tfrac{1}{2}\pi\int d\sigma\int_0^{2\pi}rd\phi. \qquad (30)$$

This integral is related to the self-potential of a substance uniformly distributed over the section. In fact, let ρ be the distance between two elements $d\sigma$ and $d\sigma'$ placed at the points P and P' (Fig. 5). Then the potential at P will be

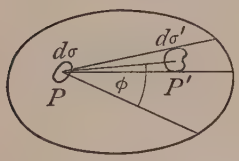

Fig. 5.

$$\int\frac{d\sigma'}{\rho}=\int_0^{2\pi}\int_0^r d\phi d\rho=\int_0^{2\pi}rd\phi,$$

and if this be integrated over $d\sigma$, we have

$$\int d\sigma\int_0^{2\pi}rd\phi.$$

For a tube of circular section the integral is easily evaluated.

Let O (Fig. 6) be the centre of the circle and OA a fixed direction, so that $\angle APQ_0=\phi$. Put $OP=l$ and $OA=a$. Then

$$Q_0Q_0'=2\sqrt{a^2-l^2\sin^2\phi}.$$

In (30) r can be replaced by Q_0Q_0', provided we integrate over ϕ from 0 to π. Thus we find

$$G=\pi\int d\sigma\int_0^{\pi}\sqrt{a^2-l^2\sin^2\phi}\,d\phi.$$

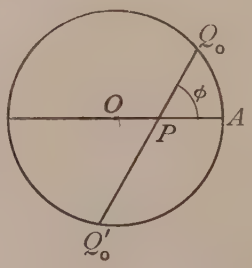

Fig. 6.

Now, taking for $d\sigma$ a ring of area $2\pi ldl$, we find

$$G=4\pi^2\int_0^a ldl\int_0^{\pi/2}\sqrt{a^2-l^2\sin^2\phi}\,d\phi.$$

Inverting the order of integrations we have

$$G = 4\pi^2 \int_0^{\pi/2} d\phi \int_0^a \sqrt{a^2 - l^2 \sin^2 \phi}\, l\, dl$$

$$= \tfrac{4}{3}\pi^2 a^3 \int_0^{\pi/2} \frac{1 - \cos^3 \phi}{\sin^2 \phi}\, d\phi = \tfrac{4}{3}\pi^2 a^3 \int_0^{\pi/2} \left\{ \frac{1 - \cos \phi}{\sin^2 \phi} + \frac{\cos \phi \sin^2 \phi}{\sin^2 \phi} \right\} d\phi$$

$$= \tfrac{4}{3}\pi^2 a^3 \int_0^{\pi/2} \left\{ \frac{1}{2 \cos^2 \frac{\phi}{2}} + \cos \phi \right\} d\phi = \tfrac{4}{3}\pi^2 a^3 \left| \tan \frac{\phi}{2} + \sin \phi \right|_0^{\pi/2} = \tfrac{8}{3}\pi^2 a^3.$$

Thus the total current across the circular section becomes

$$-\tfrac{8}{3}\pi^2 a^3 \frac{dA}{dx},$$

where

$$A = \frac{p}{m\pi} \sqrt{\frac{h}{\pi}}.$$

If u^2 be the mean squared velocity, then

$$h = \frac{3}{2u^2},$$

and since the temperature is assumed to be throughout the same, we find for the current

$$-\frac{8}{3} \sqrt{\frac{3\pi}{2}}\, \frac{a^3}{mu}\, \frac{dp}{dx}.$$

Here the current is expressed by the number of molecules. In order to express it in mass units, the last expression has to be multiplied by the mass m of a molecule. This gives

$$-\frac{8}{3} \sqrt{\frac{3\pi}{2}}\, \frac{a^3}{u}\, \frac{dp}{dx}.$$

The volume of the gas flowing across the section per unit time, at the given temperature and pressure, will be equal to the last expression divided by the density ρ, *i.e.*

$$-\frac{8}{3} \sqrt{\frac{3\pi}{2}}\, \frac{a^3}{\rho u}\, \frac{dp}{dx}. \quad . \quad . \quad . \quad . \quad (31)$$

This expression can be compared with that for the current which

would follow by applying Poiseuille's law to these small densities.* This would be, in terms of the volume,

$$-\frac{\pi a^4}{8\mu}\frac{dp}{dx},$$

where the friction coefficient μ is approximately $0{\cdot}31\rho u\lambda$,† if ρ be the density, u the root of the mean squared velocity, and λ the mean free path of the molecules. This gives for the current according to Poiseuille's law

$$-\frac{\pi a^4}{2{\cdot}4\rho u\lambda}\frac{dp}{dx}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Apart from the numerical coefficients, the currents (31) and (32) are to each other as λ to a, that is to say, as the mean free path of the molecules to the radius of the circular section of the tube.

Since in our case λ is very large compared with a, the amount of gas flowing through the section per unit time will be much greater than according to Poiseuille's law. In fact, it appears from Knudsen's experiments that it may well be 50,000 times as large.

Knudsen gives the current in terms of volume, at a pressure of 1 dyne per cm.^{2} and a temperature τ° of the gas. If ρ_1 be the density of the gas in these conditions, the current is expressed by

$$-\frac{8}{3}\sqrt{\frac{3\pi}{2}}\frac{a^3}{\rho_1 u}\frac{dp}{dx}.$$

The relation $p=\frac{1}{3}\rho u^2$ gives in the present case $\frac{1}{3}\rho_1 u^2=1$ or $u=\sqrt{3/\rho_1}$. Thus, if ρ_0 be the density at the pressure 1 and the temperature 0°, and therefore $\rho_1=\dfrac{\rho_0}{1+a\tau}$ $(a=1/273)$, we have ultimately the current

$$-C\frac{dp}{dx},$$

where

$$C=\frac{8}{3}\sqrt{\frac{\pi}{2}}\frac{a^3\sqrt{1+a\tau}}{\sqrt{\rho_0}}.$$

---

* Cf. p. 81, equation (7).

† Cf. Maxwell, *Phil. Mag.* (4), vol. xix., 1860, p. 31.

13. Knudsen's Experimental Investigation on the Flow of a Rarefied Gas through a Narrow Tube

In Knudsen's investigation this formula was tested experimentally. In his experiments the stream passed through a tube joining two vessels filled with a rarefied gas. Each vessel was connected with a MacLeod manometer by means of which it was possible to measure very small pressures by compressing the gas to a known small volume, measuring its pressure and thence calculating the original pressure by Boyle's law. For pressures smaller than 5 mm. the pressure in each vessel is in this way measured separately, while for greater pressures the pressure difference between the two vessels is measured directly.

Let the volumes of the vessels be v_1, v_2 and the pressures within them p_1 and p_2, and let $p_1 > p_2$. For a cylindrical tube of length l

$$\frac{dp}{dx} = \frac{p_2 - p_1}{l},$$

whence the current

$$-C\frac{dp}{dx} = C\frac{p_1 - p_2}{l}.$$

The volume of gas flowing through a cross-section of the tube per second is, when measured at the pressure 1,

$$-v_1\frac{dp_1}{dt} = v_2\frac{dp_2}{dt},$$

these being the quantities leaving, per second, the first and streaming into the second vessel, so that

$$-v_1\frac{dp_1}{dt} = C\frac{p_1 - p_2}{l},\quad v_2\frac{dp_2}{dt} = C\frac{p_1 - p_2}{l}.$$

Whence

$$\frac{d(p_1 - p_2)}{dt} = -(p_1 - p_2)\frac{C}{l}\left(\frac{1}{v_1} + \frac{1}{v_2}\right)$$

or

$$\log(p_1 - p_2) = -\frac{C}{l}\left(\frac{1}{v_1} + \frac{1}{v_2}\right)t + \text{const.},$$

and

$$p_1 - p_2 = \text{const.}\, e^{-\frac{C}{l}\left(\frac{1}{v_1} + \frac{1}{v_2}\right)t}.$$

Since

$$C = -l\frac{v_1 v_2}{v_1 + v_2}\frac{\Delta \log (p_1 - p_2)}{\Delta t},$$

the constant C can be determined from the change of the pressure difference during a given time.

Knudsen determined from his observations the quotient

$$C' = C/l$$

whose theoretical value is

$$\frac{8}{3}\sqrt{\frac{\pi}{2}}\frac{a^3\sqrt{1+a\tau}}{l\sqrt{\rho_0}}. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

In his experiments the value of λ/a mounted up to 6000. He worked with hydrogen, oxygen, and carbon dioxide.

In order to find the effect of *the length* of the tube, different tubes were used, of which the first was about 6 cm. long and about 0·02 cm. in diameter, while the second was about twice as long and had the same diameter. The gas in this case was hydrogen.

The ratio of the observed values of C' thus found was 1·95, while according to formula (33) it should be 2·05.

Next, to find the effect of *the cross-section*, carbon dioxide was passed through the first and through a third tube whose length was about twice and diameter about 1·4 times that of the first tube. The ratio of the experimental values of C' was 1·20, while the formula gave 1·15. Whence it appears that the current is actually inversely proportional to l and directly proportional to a^3.

In order to investigate the effect of *the density* of the gas, a set of 24 parallel tubes were used, each 2 cm. long and about 0·006 cm. in diameter.

According to the formula C' should be inversely proportional to $\sqrt{\rho_0}$ and, therefore, $C'\sqrt{\rho_0}$ should be independent of the kind of the gas. That such is actually the case, will be seen from the following results :

Gas	C' (obs.)	$\sqrt{\rho_0}$	$C'\sqrt{\rho_0}$
Hydrogen . . .	0·168	1	0·168
Oxygen . . .	0·0409	4	0·164
Carbon dioxide . .	0·0348	4·69	0·163

To find the influence of the *temperature* of the gas, Knudsen made use of the first tube and filled the vessels with hydrogen at a mean pressure of 0·3 mm. In the results of the experiments the gas volume flowing through the tube is reduced to the temperature 0°, so that

$$C' = \frac{8}{3}\sqrt{\frac{\pi}{2}}\,\frac{a^3}{l\sqrt{\rho_0(1+\alpha\tau)}}.$$

Consequently,

$$C'\sqrt{1+\alpha\tau}$$

should, according to the theory, be independent of the temperature. Now, Knudsen found :

Temperature	C'	$C'\sqrt{\frac{1+\alpha\tau}{1+22a}}$
22°	0·0713	0·0713
100°	0·0641	0·0721
196°	0·0588	0·0741

This table shows that also the dependence of C' on the temperature is correctly represented by the formula.

Finally Knudsen proved the correctness of the *constant* factor in the formula by actually determining in different cases the absolute amount of the gas streaming through and by comparing the results with those calculated by the formula in question. His results were

Tubes	Gas	C' (obs.)	C' (calc.)
1	Hydrogen . .	0·073	0·080
1	Oxygen . . .	0·0187	0·0202
1	Carbon dioxide .	0·0166	0·0172
2	Hydrogen . .	0·0375	0·0392
3	Carbon dioxide .	0·0199	0·0198
4	Hydrogen . .	0·168	0·161
4	Oxygen . . .	0·0409	0·0404
4	Carbon dioxide .	0·0348	0·0344

With reference to this very rarefied state of the gas Knudsen speaks of a " pure molecular streaming." He extends also his researches to the case of larger densities for which the collisions of the molecules can no longer be disregarded and which offers considerable theoretical difficulties. In the latter case Knudsen speaks of " mixed molecular streaming."

14. Flow of a Rarefied Gas through a Narrow Orifice; Knudsen's Experiments

A second investigation due to Knudsen* concerns *the flow of a gas through a narrow orifice.*

We assume again that the gas is very strongly rarefied, so that the mutual collisions of the molecules can be disregarded, and that the dimensions of the orifice are small compared with the free path of the molecules. We suppose, further, that the temperature of the gas is throughout the same and that the diaphragm with the said orifice divides the space occupied by the gas in two portions, so that the pressure above the diaphragm is throughout p_1, and below the diaphragm p_2.

By what was said before the number of particles passing, per unit time, through an element $d\sigma$ of the orifice from the upper to the lower chamber in a direction contained within a cone of angular aperture $d\omega$ can be written

$$A_1 \cos\theta \, d\omega d\sigma,$$

where θ is the angle between the axis of the cone and the normal of the surface element, and A_1 a factor depending on the temperature of the walls of the upper chamber and on the pressure of the gas therein contained, but independent of the cone chosen. Thus the number of particles flying downwards through $d\sigma$ per unit time will be

$$\int_0^{\pi/2} 2\pi A_1 \cos\theta \sin\theta d\theta d\sigma = \pi A_1 d\sigma,$$

and therefore the total number through the whole orifice Σ,

$$\pi A_1 \Sigma.$$

Similarly the number of particles flying, per unit time, upwards is

$$\pi A_2 \Sigma.$$

Whence, the resultant flux downwards, measured by the number of molecules,

$$\pi(A_1 - A_2)\Sigma.$$

Since

$$A = \frac{p}{m\pi}\sqrt{\frac{h}{\pi}}, \; h = \frac{3}{2u^2},$$

* *Ann. der Phys.*, xxviii., 1909, p. 999.

the last expression becomes

$$\frac{1}{mu}\sqrt{\frac{3}{2\pi}}(p_1-p_2)\Sigma.$$

As before, Kundsen expresses the flux through the orifice per unit time in terms of the volume the gas would occupy at a temperature $\tau°$ and a pressure of one dyne per cm.^{2}. If ρ_1 be in this case the density, then $u=\sqrt{3}/\rho_1$ (cf. p. 116), and the expression for the flux becomes

$$\sqrt{\frac{1}{2\pi\rho_1}}(p_1-p_2)\Sigma.$$

This formula was again tested experimentally by Knudsen who used the same apparatus as in the preceding investigations, but having replaced the original tube by a glass tube containing a plate of platinum in which an orifice was pierced by means of a fine needle. He worked with orifices of an area of 0·0005 mm.^{2}. According to the last formula the flux is proportional to p_1-p_2 and can thus be written

$$C(p_1-p_2),$$

the theoretical value of C being $\Sigma/\sqrt{2\pi\rho_1}$. The experimental determination of C proceeds again on the same lines as in the previous experiments. In fact, if v_1 and v_2 be the volumes of the gas at the two sides of the orifice, the volume of the gas streaming through per unit time, referred to the pressure of 1 dyne per cm.^{2}, is

$$-v_1\frac{dp_1}{dt}=v_2\frac{dp_2}{dt}=C(p_1-p_2),$$

whence

$$C=-\frac{v_1v_2}{v_1+v_2}\frac{\Delta\log(p_1-p_2)}{\Delta t}.$$

The following are some of Knudsen's results :

Gas	C (calculated from observations)	C (by theoretical formula)
Hydrogen . . .	0·225	0·236
Oxygen	0·0565	0·0576
Carbon dioxide . .	0·0465	0·0491

These results give, in Knudsen's words, an experimental confirmation of the correctness of the kinetical theory of gases,

and especially of the validity of Maxwell's law of the distribution of velocities. If one assumed against Maxwell's theory that all molecules have the same velocity, one would find from the theoretical formula for the coefficient C values by 8·6 per cent. greater than those of the above table. The observed values of C would then differ from the theoretical ones by amounts which could not be thrown upon the experimental errors.

Also in the case of a gas flowing through a narrow orifice the phenomena are different at a higher pressure and their theory becomes difficult. We have then to distinguish between the case in which the mean free path is comparable with the dimensions of the orifice and that in which it is small compared with the latter, which finally leads to the formation of jets.

15. Flow of a Rarefied Gas through a Narrow Tube whose Ends have Different Temperatures

We will now return to *the flow of a gas through a tube* in order to investigate the effect of *temperature differences* at different places along the tube.

By what was said before (p. 115), the current is determined by

$$-\frac{8}{3}\pi^2 a^3 \frac{dA}{dx},$$

where

$$A = \frac{p}{mu}\sqrt{\frac{3}{2\pi^3}},$$

and it can therefore be written

$$-\frac{8}{3}\frac{a^3}{m}\sqrt{\frac{3\pi}{2}}\frac{d}{dx}\left(\frac{p}{u}\right).$$

The current is thus dependent on the pressure gradient and the temperature gradient. Consequently, the condition for the absence of streaming in the tube is

$$\frac{p}{u} = \text{const.},$$

or

$$\frac{p}{\sqrt{T}} = \text{const.}$$

Thus we see that, while in a wide tube, in comparison with

whose dimensions the mean free path of the molecules is small, the condition of equilibrium is that the pressure should be everywhere the same and, therefore, the density inversely proportional with the absolute temperature T (since $\frac{pv}{T} = \text{const.}$), in the present case the pressure must be proportional to $\sqrt{T}$ and, therefore, the density inversely proportional to $\sqrt{T}$.

In a narrow tube whose extremities have different temperatures these pressure and density differences will arise by themselves. In fact, if originally the pressure is everywhere the same, the gas will stream, according to the above formula, towards places of highest temperature, so that the pressure will increase there.

This was verified by Knudsen * experimentally. He connected two MacLeod manometers by a set of tubes of different cross-sections. In his first experiments a single capillary tube only was used, 9 cm. long and 0·6 mm. in diameter. This capillary was placed between two tubes of 14 mm. in diameter. The junction of one of the wide tubes with the capillary was brought to a temperature of about 350°, which raised the other end of the capillary to a temperature of about 100°. The determination of these temperatures, however, was uncertain. The two manometers could also be connected with each other directly by a wide tube. The purpose of the observations with a direct connection, when the pressure in the two manometers should be found equal, was only to form an idea of the accuracy of the measurements. The results obtained for hydrogen were :

	p_1	p_2
For direct connection . . .	0·0218	0·0216
For connection by capillary tube .	0·0223	0·0211

There is thus, in fact, a pressure difference between the two ends of the capillary tube. The whole system shows much similarity to a thermo-element. If the two wide tubes are, besides the capillary, connected also by a wide tube, the gas will continue to flow, and in such a direction that the stream in the capillary is from a lower towards a higher temperature. This stream is thus analogous to a thermo-electric current. The effect

* *Ann. der Phys.*, xxxi., 1910, p. 205.

will also here be increased by connecting the manometers by a large number of wide and of narrow tubes, alternately arranged (Fig. 7). At the places marked in the figure by rectangles the tubes were heated by means of platinum wires which were brought to glowing by an electric current. The temperature difference between the heated and the not-heated contact places of the tubes was measured by means of a thermo-element and amounted to about 500°. The following were the results for hydrogen :

p_1	p_2	$\frac{p_1}{p_2}$ (observed)	$\frac{p_1}{p_2}$ (calculated theoretically)
0·00978	0·00419	2·33	2·49

The pressure difference for different temperatures in different parts of a tube filled with a strongly rarefied gas was demonstrated by Knudsen in yet another way.* In a glass tube of

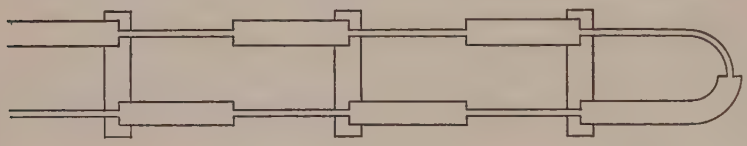

FIG. 7.

7·5 mm. in diameter magnesium powder is placed between asbestum stoppers. One end of the magnesium column is heated electrically up to a temperature of 248°, which gives to the other end a temperature of 22°. The pressures at the two ends of the tube are measured by means of MacLeod manometers. The equilibrium in this case sets in very slowly. Finally we should have

$$\frac{p_1}{p_2} = \sqrt{\frac{T_1}{T_2}},$$

where p_1 and p_2 are the measured pressures.

Knudsen found for p_1 and p_2, after different times :

Time	p_1	p_2
0	0·3000	0·3000
18h 25′	0·3308	0·2716
23h 25′	0·3347	0·2693
46h 50′	0·3414	0·2641
∞	0·3428	0·2628

The last values of p_1 and p_2 which would correspond to a perfect

* *Ann. der Phys.*, xxxi., 1910, p. 633.

equilibrium were obtained by extrapolation, it being assumed that the relation

$$\frac{d(p_1 - p_2)}{dt} = -\alpha[(p_1 - p_2) - q]$$

is valid, q being the pressure difference in the state of equilibrium.

From what precedes it is clear that a porous plate should show the following effect. If one face of the plate is warmer than the other, air should stream from the colder to the warmer face, and this flow should continue so long as there is a temperature difference between the two faces.

This phenomenon was demonstrated by Knudsen * in a simple way. In a vessel of porous material air is heated electrically. Thus the inner surface of the vessel is brought to a higher temperature than its outer surface, and air is being drawn through the walls into the vessel. If, therefore, the vessel be fitted with a tube ending in a bottle containing water, air will be seen to bubble up through the water. Knudsen found that through a vessel of the capacity of 100 cm.^{3} as many cubic centimetres of air could easily be drawn. If the air cannot escape, a pressure difference is produced. With a strong heating the pressure within the vessel is a few centimetres (of mercury) higher than outside.

Knudsen points out that such phenomena should happen often, and that also in the case of greater gas densities a temperature difference will bring about a pressure difference. According to Knudsen this phenomenon is undoubtedly of great importance in nature. It plays perhaps a rôle in the respiration of plants and contributes to the refreshing of air in the porous soil, air being given out where the surface of the earth is heated by the sun, and sucked in where no such heating takes place.

16. Mutual Repulsion of Two Plates at Different Temperatures separated by a Rarefied Gas

Knudsen † has constructed an *absolute manometer* based upon radiometric action. This was suggested to him by an investigation on the change of the velocity of gas molecules due to their impact against a fixed wall differing in temperature from the

* *Ann. der Phys.*, xxxi., 1910, p. 207.
† *Ibid.*, xxxii., 1910, p. 809.

gas. This occurs already in Crookes' radiometer. In this apparatus four vanes of mica, covered on one side with lampblack, are mounted on the arms of a horizontal cross free to spin round on a pivot. If the mutual collisions of the air molecules are disregarded, the following explanation of the motion of the radiometer seems to be the simplest. Both faces of each vane are hit by equally numerous molecules and in like manner, but those molecules which impinge against the blackened faces will be thrown back with a greater velocity than those hitting the clear faces (for, owing to a greater absorption, the former have a higher temperature). As a result the vanes will revolve with their unblackened faces turned forward. To many physicists, however, it seemed doubtful whether this simple theory could be here applied.

Now, Knudsen considers a case for which the theory can be adequately developed. He takes two plates S and S' (Fig. 8) immersed in a strongly rarefied gas and placed at a distance from each other which is small in comparison with the mean free path of the molecules. The plate S has the temperature of the surrounding medium, while S' is brought to a higher temperature, so that the plates will exert upon each other a repelling force. This force is a function of the pressure. The latter can therefore be determined by measuring the former, which gives the principle for constructing a manometer.

We saw from the preceding discussion of the flow of a gas through narrow tubes and small orifices that a temperature difference gives rise to a pressure difference. Let p be the pressure between the plates, p_0 the pressure outside, and T and T' the temperatures of the two plates. Then Knudsen takes for the temperature between the plates the mean $\frac{1}{2}(T+T')$ and derives the relation

$$\frac{p}{p_0}=\frac{\sqrt{\frac{1}{2}(T+T')}}{\sqrt{T}}.$$

If K be the repelling force per unit area, this relation gives

$$K=p-p_0=\left[\sqrt{\frac{T+T'}{2T}}-1\right]p_0$$

and

$$p_0=\frac{K}{\sqrt{\frac{T+T'}{2T}}-1}. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The formula used by Knudsen in his experiments with the manometer differed from this somewhat; it was

$$p_0 = \frac{2K}{\sqrt{\frac{T'}{T}} - 1}. \quad . \quad . \quad . \quad . \quad . \quad (34')$$

The force K is measured by means of a torsion balance to which the plate S is attached, while the fixed plate S' is heated electrically.

Knudsen subjects the formula (34') to a further scrutiny, but we shall not follow him in this and shall consider the question in a somewhat different way.

We assume that the dimensions of the plates are large compared with their distance, that the molecules move from one plate to the other in all directions according to the cos θ-rule (cf. p. 110), and that Maxwell's velocity distribution holds.

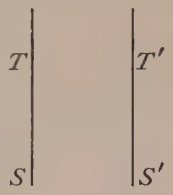

FIG. 8.

Let us now follow a molecule on its path through a unit time, during which interval it moves many times back and forth, and as many times towards the right as towards the left. We will calculate the number of flights in a certain direction, say, towards the right, which we will denote by ν. The number of flights for which the velocity of the molecule is contained between v and $v+dv$ and the angle of inclination of this velocity to the normal erected upon the plates between θ and $\theta+d\theta$, will be found by multiplying the number ν by the probability that v and θ should fall within these limits, that is to say, by the fraction of all molecules rebounding, per unit time, from one of the plates, for which the velocity and direction of motion satisfy these conditions.

The number of molecules, per unit volume, whose velocities lie between v and $v+dv$ is proportional to $e^{-hv^2}v^2dv$, and the number of all such molecules leaving per unit time one of the plates is equal to

$$Ce^{-hv^2}v^3dv,$$

where h corresponds to the temperature of that plate. The constant C must satisfy the condition

$$C\int_0^\infty e^{-hv^2}v^3dv = 1,$$

whence

$$C = 2h^2.$$

The requirement has still to be satisfied that the direction of motion should make with the plate normal an angle included between θ and $\theta+d\theta$. The number of particles leaving the plate is, with equal cone aperture, assumed to be proportional to $\cos\theta$. Again, the (angular) space between two cones of apertures θ and $\theta+d\theta$ is proportional to $\sin\theta d\theta$. Thus, the probability that the molecules will have the required direction of motion,

$$C'\sin\theta\cos\theta d\theta,$$

where C' is determined by the condition

$$C'\int_0^{\pi/2}\sin\theta\cos\theta d\theta=1,$$

or

$$C'=2.$$

Since these two probabilities are independent of each other, the required probability will be equal to their product, *i.e.* to

$$4h^2\sin\theta\cos\theta e^{-hv^2}v^3d\theta dv.$$

Multiplying this by ν we shall find the number of passages towards the right made by a molecule per unit time under the said conditions. The same expression will be found for the corresponding number of passages towards the left, provided h is replaced by h', where h' belongs to the temperature T' of the heated plate.

An expression for the number ν itself will now be found by noticing that the time taken by a single passage is $l/(v\cos\theta)$, where l is the mutual distance of the plates, and that the time taken by all the passages under consideration is the unit of time. Thus,

$$4l\nu\left\{\int_0^{\pi/2}\sin\theta d\theta\int_0^{\infty}h^2e^{-hv^2}v^2dv+\int_0^{\pi/2}\sin\theta d\theta\int_0^{\infty}h'^2e^{-h'v^2}v^2dv\right\}=1.$$

Since

$$\int_0^{\infty}e^{-hv^2}v^2dv=\frac{1}{4}\sqrt{\frac{\pi}{h^3}},$$

this gives

$$\nu=\frac{1}{l\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}.$$

In order to find the pressure between the plates we have only to take the difference of the components along the plate normal of the momentum of all the molecules flying away, per unit of

time, from the plate S and of all those flying towards the plate S. These two components have opposite signs. Consequently, the pressure will be given by the sum of their absolute values.

For the number of times a molecule, within the given limits of v and θ, rebounds from the plate S, we have found, per unit of time,

$$4\nu h^2 \sin\theta \cos\theta e^{-hv^2} v^3 d\theta dv. \quad . \quad . \quad . \quad (35)$$

With this direction of motion and this velocity the component of momentum along the plate normal is $mv\cos\theta$. Thus, multiplying the expression (35) by $mv\cos\theta$ and integrating over all values of v and θ, we shall find the momentum carried by a particle which rebounds from S, per unit time. Multiplying this further by the total number of particles, we shall have the pressure exerted upon the plate by the molecules which move towards the right. For the pressure due to the molecules moving towards the left a similar expression holds, with h replaced by h'. Dividing by the area of the plate, we shall have the pressure per unit area. Thus, if n be the number of molecules per unit volume and l the distance of the plates, the pressure will be

$$p = 4\nu mnl\left[\int_0^{\pi/2} \sin\theta\cos^2\theta d\theta \int_0^\infty h^2 e^{-hv^2} v^4 dv + \int_0^{\pi/2} \sin\theta\cos^2\theta d\theta \int_0^\infty h'^2 e^{-h'v^2} v^4 dv\right].$$

Since

$$\int_0^\infty e^{-hv^2} v^4 dv = \frac{3}{8}\sqrt{\frac{\pi}{h^5}},$$

we have ultimately

$$p = \tfrac{1}{2} m\nu nl\sqrt{\pi}\left(\frac{1}{\sqrt{h}} + \frac{1}{\sqrt{h'}}\right) = \frac{mn}{2\sqrt{hh'}}.$$

In order to verify this result we can put $h = h'$. Since $h = \frac{3}{2u^2}$, we have

$$p = \frac{mn}{2h} = \frac{1}{3} mnu^2,$$

which is the correct value of the pressure when both plates have the same temperature.

Thus the pressure in the space between the plates is determined. The pressure p_0 outside the plates is smaller, and the difference $p - p_0$ gives the repelling force K.

The temperature is throughout the greater part of the space outside the plates equal T, and the density is only in a thin layer adjacent to the plate S' smaller than in the remaining space. If this thin layer be disregarded, the number of molecules per unit volume can be put equal n_0 and the pressure

$$p_0 = \frac{mn_0}{2h}.$$

Thus we find

$$K = \frac{mn}{2\sqrt{hh'}} - \frac{mn_0}{2h} = p\left(\frac{n}{n_0}\sqrt{\frac{h}{h'}} - 1\right). \quad . \quad . \quad (36)$$

It remains to establish a relation between n and n_0. For this purpose suppose first both the plates had the same temperature T. Then the density of the gas would be everywhere the same. If now the plate S' is heated up to the temperature T', the first consequence will be that at the edge of the plates more particles will pass from the space between the plates into the outer space than *vice versa*, until an equilibrium is reached and as many particles pass in one as in the other direction. A relation between n and n_0 will thus be found by equating to each other the numbers of particles moving in one and in the opposite direction. Since, however, the state of motion near the edges is complicated, we will simplify the reasoning by an artifice, viz. by supposing that there is an orifice in the plate S which is so small as not to change the state perceptibly. Then, in the case of equilibrium, also the number of molecules passing through the orifice into the space between the plates will be equal to the number of those passing through the orifice outwards. Let ω be the area of the orifice. In the first place, the number of particles passing through the orifice outwards, per unit time, is equal to the number of those which per unit time strike the area ω of the plate S. By what precedes, this number is

$$\nu n\omega l = \frac{\omega n}{\sqrt{\pi}(\sqrt{h} + \sqrt{h'})}.$$

Next, to find the number of particles passing per unit time inside, notice that the orifice is made in the non-heated plate, so that in the space outside the plates and also at the surface of the plate S the temperature can be taken to be T and the number of particles per unit volume n_0. The number of particles per

unit volume having a velocity between v and $v+dv$ and a direction of motion between θ and $\theta+d\theta$ is $Cn_0 \sin\theta e^{-hv^2}v^2 d\theta dv$, where C is determined by the condition

$$Cn_0\int_0^{\pi}\sin\theta d\theta\int_0^{\infty}e^{-hv^2}v^2dv = n_0$$

which gives

$$C = 2\sqrt{\frac{h^3}{\pi}}.$$

The number of particles per unit volume satisfying the said condition for v and θ is, therefore,

$$2\sqrt{\frac{h^3}{\pi}}\,n_0 \sin\theta e^{-hv^2}v^2 d\theta dv. \quad . \quad . \quad . \quad (37)$$

The number of particles striking, per unit time, the area ω will be the product of (37) into $v\omega\cos\theta$ integrated over all v and over θ from 0 to $\frac{1}{2}\pi$, *i.e.*

$$2\sqrt{\frac{h^3}{\pi}}\,n_0\omega\int_0^{\pi/2}\sin\theta\cos\theta d\theta\int_0^{\infty}e^{-hv^2}v^3dv = \frac{\omega n_0}{2\sqrt{\pi h}}.$$

Equating this number to that found above for the number of particles passing through the orifice towards the left, we find

$$\frac{\omega n}{\sqrt{\pi}(\sqrt{h}+\sqrt{h'})} = \frac{\omega n_0}{2\sqrt{\pi h}},$$

or

$$\frac{n}{n_0} = \frac{\sqrt{h}+\sqrt{h'}}{2\sqrt{h}}.$$

Substituting this into (36), we have

$$K = p_0\left(\frac{\sqrt{h}+\sqrt{h'}}{2\sqrt{h'}} - 1\right) = \tfrac{1}{2}p_0\left(\sqrt{\frac{h}{h'}} - 1\right),$$

or, since $h/h' = T'/T$,

$$K = \tfrac{1}{2}p_0\left(\sqrt{\frac{T'}{T}} - 1\right), \quad . \quad . \quad . \quad . \quad (38)$$

which is the formula used by Knudsen (cf. (34′), p. 127).

If $T'-T$ is small, this expression, as well as (34), reduces to

$$K = p_0\,\frac{T'-T}{4T}. \quad . \quad . \quad . \quad . \quad (39)$$

Notice that in deducing (38) it was assumed that every particle after striking a fixed wall acquires a velocity which corresponds to the temperature of the wall. From Knudsen's later investigations it appears that this is not the case, and that the mean variation of the kinetic energy is only a fraction of what it would be according to our assumption.

17. Knudsen's Manometers

Upon the formula

$$p_0 = \frac{2K}{\sqrt{\frac{T'}{T}} - 1}$$

is based the use of Knudsen's manometer. Here p_0 is the total pressure, including that of the mercury vapour. The formula holds for $p_0 < {}_{3\,0\,0}^{1}$ mm. of mercury (4 to 5 dynes per cm.^{2}). For greater pressures we can take

$$p_0 = c\frac{2K}{\sqrt{\frac{T'}{T}} - 1},$$

where $c > 1$. The dependence of p_0 upon the temperature remains as in the original formula, since c is independent of temperature.

The first apparatus used by Knudsen consisted of a platinum strip S' which was fixed and could be heated electrically, the temperature being determined by measuring the resistance. At a short distance from this was placed a platinum plate suspended on the arm of a torsion balance by means of which the force K could be measured. The whole apparatus was placed under a jar in communication with a Gaede mercury pump.

The following are the results of some experiments in which the plates were placed in air :

T	31·7	118·9	198·5	274·5	376·2
T'	23·4	23·7	24·7	28·0	37·2
p_0 in dynes per cm.^{2} (calculated from K) . .	2·31	2·45	2·27	2·21	2·31

Since the pressure between the plates was kept constant as far as possible, these numbers give a good verification of the formula.

The mean of the p_0-values is 2·28 dynes per cm.², while the measurement with a MacLeod manometer gives 0·20 dyne per cm.². To the latter must be added the pressure of the saturated mercury vapour, for which Knudsen's previous measurements,* at a temperature of about 23°, gave 2·04 dynes per cm.². Thus the total pressure becomes 2·24 dynes per cm.², which agrees well with the pressure found with the new apparatus. Knudsen has repeated these measurements with hydrogen.

The condition that the plate distance should be small compared with the mean free path λ of the molecules was satisfied. In fact, since at a pressure of 1 dyne per cm.² the mean free path λ amounts to about 10 cm., in the experiments under discussion λ was about 4 cm., while the distance of the plates amounted only to 0·055 cm.

Other forms of manometers are described in the quoted paper by Knudsen. One of these apparatus consisted of a copper cylinder with a polished end-surface which served as the heated plate. The cylinder is heated by an electric current, and the temperature is measured by means of a mercury thermometer placed within a cavity of the cylinder. Opposite the polished end-surface of the cylinder there is a copper plate attached to the bulb of a thermometer which is suspended on the arm of a torsion balance. The copper cylinder is surrounded by sheets of copper which are not heated, and serve as guard rings. By means of a special system of pipettes Knudsen was able to introduce into the testing space exceedingly small quantities of gas. For measurements at high temperatures he used an apparatus of platinum.

Even at very high temperatures the formula (38) turned out to agree fairly well with the experimental results, while this did not seem to be the case with the approximate formula (39).

Yet another very simple apparatus consisted of a wide glass tube A (Fig. 9), into which a second glass tube b was sealed. The tube b had on one side, at c, an aperture. Opposite this, in the middle of the tube, was suspended on two cocoon threads a plate of mica, P. When the apparatus is placed in warm water, the plate of mica experiences

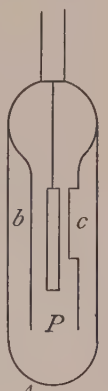

FIG. 9.

* *Ann. der Phys.*, xxix., 1909, p. 179.

a repulsion, due to the difference of its own temperature and of that of the outer wall. From the positions assumed by this plate, when the apparatus was placed in cold and then in warm water, the repulsion, and thence also the pressure of the gas, could be measured. After a certain time, when also the innermost tube reached the temperature of the bath, the repulsion ceased.*

18. Knudsen's Accommodation Coefficient

In a further paper, entitled "The Molecular Heat Conduction of Gases and the Accommodation Coefficient,"† Knudsen points out that the velocities with which the molecules, impinging upon a fixed wall, rebound from it do not correspond to the temperature of the wall. He introduces, therefore, an accommodation coefficient a, by which the temperature variation which the molecules would undergo at the impact, if they assumed the wall temperature, has to be multiplied in order to give the true temperature variation. The accommodation coefficient depends on the nature of the gas and of the wall. The more rough the wall, the greater the value of a, tending to 1.

Let us once more consider the two plates S and S' (Fig. 8) at the temperatures T and T'. Then the molecules which fly away from S towards the right will not have, as previously assumed, the temperature T but some other temperature θ, and similarly those moving away from S' towards the left will have a temperature θ' instead of T', where θ and θ' lie between T and T'.

By the definition of the accommodation coefficient we have the relations

$$\theta' - \theta = a(\theta' - T),$$

$$\theta - \theta' = a(\theta - T'),$$

whence

$$\theta = T + \frac{1-a}{2-a}(T' - T),$$

$$\theta' = T' + \frac{1-a}{2-a}(T - T'),$$

$$\theta' - \theta = \frac{a}{2-a}(T' - T).$$

* For the sake of completeness it may be mentioned that Langmuir (*Phys. Rev.* (2), 1, 1913, p. 337) constructed a manometer based on the properties of rarefied gases, by means of which pressures as small as 10^{-7} mm. of mercury could be measured.

† *Ann. der Phys.*, xxxiv., 1911, p. 593.

The accommodation coefficient is here assumed equal for the two plates. In calculating the pressure the velocities of the molecules can be taken as if $a=1$, provided that T and T' are replaced by θ and θ'.

Whence it follows that in the expressions

$$p=\frac{mn}{2\sqrt{hh'}} \text{ and } p_0=\frac{mn_0}{2h_0},$$

h_0 corresponds to the temperature T, while h and h' correspond to the temperatures θ and θ' respectively.

In order to determine the force K we again assume the presence of a small orifice ω in the plate S and introduce the condition that, per unit time, as many molecules should pass through ω into the space between the plates as cross it outwards. This gives the equation

$$\frac{\omega n}{\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}=\frac{\omega n_0}{2\sqrt{\pi h_0}}.$$

Thus,

$$\frac{n}{n_0}=\frac{\sqrt{h}+\sqrt{h'}}{2\sqrt{h_0}},$$

and

$$K=p-p_0=p_0\left\{\frac{\sqrt{h_0}}{2}\left(\frac{1}{\sqrt{h}}+\frac{1}{\sqrt{h'}}\right)-1\right\}. \quad \text{(Cf. p. 131.)}$$

The circumstance that the accommodation coefficient is not equal to unity affects only the term $\frac{1}{\sqrt{h}}+\frac{1}{\sqrt{h'}}$. Since h is inversely proportional to the absolute temperature, we can put

$$h=\frac{C}{\theta}, \; h'=\frac{C}{\theta'}.$$

Now, $T+T'=\theta+\theta'$. Consequently, in the expression

$$\frac{1}{\sqrt{h}}+\frac{1}{\sqrt{h'}}=\frac{1}{\sqrt{C}}(\sqrt{\theta}+\sqrt{\theta'})$$

θ and θ' can be replaced by T and T', provided the temperature difference is small. Only for greater temperature differences must the accommodation coefficient be taken into account.

19. Heat Conduction in a Rarefied Gas contained between Two Plates of Unequal Temperature

We now pass to Knudsen's investigation on *heat conduction* * and, for this purpose, consider again the plates S and S' (Fig. 8). We have to calculate how much energy (for polyatomic gases including also the internal energy of the molecules) is transferred, per second, from one plate to the other. We know from previous considerations that a molecule flies away from the plate S a number ν of times per second, where

$$\nu = \frac{1}{l\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}. \quad \text{(Cf. p. 128.)}$$

The probability that the velocity of such a molecule is contained between v and $v+dv$ and that the direction of its motion makes with the plate normal an angle falling between θ and $\theta+d\theta$ was already deduced and amounts to

$$4h^2 \sin\theta \cos\theta e^{-hv^2} v^3 d\theta dv \text{ (cf. p. 129)},$$

where h is determined by the temperature of the plate from which the molecule rebounds.

We now introduce the new condition that the internal energy of the molecule should lie between ϵ and $\epsilon+d\epsilon$. Let the probability of this be represented by $f(h, \epsilon)\, d\epsilon$, where h indicates that the function f depends on the temperature. Since, by Boltzmann's distribution law, the function f is independent of v, the probability for all the three conditions to be satisfied is equal to the product of the probabilities of each of them, so that the number of times a molecule rebounds from S under the said conditions is

$$4\nu \sin\theta \cos\theta h^2 e^{-hv^2} v^3 f(h, \epsilon) d\theta dv d\epsilon.$$

Here the function $f(h, \epsilon)$ satisfies the condition

$$\int_0^\infty f(h, \epsilon) = 1.$$

If $\overline{\epsilon_h}$ is the mean internal energy at a temperature corresponding to h,

$$\int_0^\infty \epsilon f(h, \epsilon) d\epsilon = \overline{\epsilon_h}.$$

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* *Ann. der Phys.*, xxxiv., 1911, p. 593.

Since a molecule leaving the plate carries with it the energy $\frac{1}{2}mv^2+\epsilon$, we find for the total energy transfer, per unit of time,

$$4\nu\int_0^{\pi/2}\sin\theta\cos\theta d\theta\int_0^\infty\int_0^\infty h^2e^{-hv^2}v^3(\tfrac{1}{2}mv^2+\epsilon)f(h,\epsilon)d\epsilon dv.$$

The integration over θ gives

$$2\nu h^2\int_0^\infty\int_0^\infty e^{-hv^2}v^3(\tfrac{1}{2}mv^2+\epsilon)f(h,\epsilon)d\epsilon dv,$$

and that over v,

$$2\nu h^2\int_0^\infty\left(\frac{m}{2h^3}+\frac{\epsilon}{2h^2}\right)f(h,\epsilon)d\epsilon,$$

since

$$\int_0^\infty e^{-hv^2}v^3dv=\frac{1}{2h^2},\quad\int_0^\infty e^{-hv^2}v^5dv=\frac{1}{h^3}.$$

Finally, an integration over ϵ gives

$$2\nu h^2\left(\frac{m}{2h^3}+\frac{\overline{\epsilon_h}}{2h^2}\right)=\nu\left(\frac{m}{h}+\overline{\epsilon_h}\right).$$

If n is the number of molecules per unit volume and l the distance of the plates, the flux of energy per unit area will be

$$\nu nl\left(\frac{m}{h}+\overline{\epsilon_h}\right)=\frac{n}{\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}\left(\frac{m}{h}+\overline{\epsilon_h}\right).$$

The energy gained by the plate, per unit area and unit time, from the molecules moving in the opposite direction is given by the same expression, only with h replaced by h' in the last factor.

Ultimately, therefore, we find for the heat transfer from the plate S' towards the plate S

$$\frac{n}{\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}\left(\frac{m}{h'}-\frac{m}{h}+\overline{\epsilon_{h'}}-\overline{\epsilon_h}\right).$$

Since $w^2=3/2h$, the term m/h can be replaced by $\frac{4}{3}\,\overline{\epsilon_{oh}}$, where $\overline{\epsilon_{oh}}$ is the mean kinetic energy of translatory motion at a temperature corresponding to h. Similarly m/h' will be replaced by $\frac{4}{3}\,\overline{\epsilon_{oh'}}$. The heat transfer from S' to S can thus be written

$$\frac{n}{\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}\left(\frac{4}{3}\,\overline{\epsilon_{oh'}}-\frac{4}{3}\,\overline{\epsilon_{oh}}+\overline{\epsilon_{h'}}-\overline{\epsilon_h}\right).$$

The factor $\frac{4}{3}$ shows that in the process of heat transfer the energy of translatory motion plays a greater rôle than the internal energy, which can easily be explained, since with a quicker translation the molecules move also more often to and fro between the plates. If the temperature difference $T' - T$ is small, the last expression can be written

$$\frac{n}{\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}\left(\frac{4}{3}\frac{d\bar{\epsilon}_o}{d\bar{T}}+\frac{d\bar{\epsilon}}{d\bar{T}}\right)(T'-T). \quad . \quad . \quad (40)$$

This holds also for greater temperature differences provided ϵ_0 and ϵ are linear functions of T, which is the case of ϵ_o.

The quantities $d\bar{\epsilon}_o/dT$ and $d\bar{\epsilon}/dT$ are related to the specific heats. If c_v be the specific heat at constant volume, and c_p that at constant pressure, both per unit mass, then

$$c_v = \frac{1}{m}\left(\frac{d\bar{\epsilon}_o}{dT}+\frac{d\bar{\epsilon}}{dT}\right),$$

$$c_p - c_v = \frac{2}{3m}\frac{d\bar{\epsilon}_o}{dT},$$

whence

$$c_p = \frac{1}{m}\left(\frac{5}{3}\frac{d\bar{\epsilon}_o}{dT}+\frac{d\bar{\epsilon}}{dT}\right),$$

$$c_p + c_v = \frac{2}{m}\left(\frac{4}{3}\frac{d\bar{\epsilon}_o}{dT}+\frac{d\bar{\epsilon}}{dT}\right).$$

Ultimately, therefore, the expression (40) becomes

$$\frac{mn}{2\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}(c_p+c_v)(T'-T), \quad . \quad . \quad (41)$$

which is proportional to the number of molecules per unit volume, and thus to the density.

Knudsen writes for this heat transfer per unit area

$$W = (T'-T)p\epsilon_K, \quad . \quad . \quad . \quad . \quad (42)$$

where p is the pressure (there being, for a small temperature difference, no need for distinguishing between the pressures at T and T') and ϵ_K is a coefficient introduced by Knudsen which, by (41), has the value

$$\epsilon_K = \frac{mn}{2p\sqrt{\pi}(\sqrt{h}+\sqrt{h'})}(c_p+c_v).$$

Since $h=\rho/2p$ and $mn=\rho$, while $\sqrt{h}+\sqrt{h'}$ can be replaced by $2\sqrt{h}$, this expression reduces to

$$\epsilon_K=\frac{1}{\sqrt{8\pi}}\sqrt{\frac{\rho}{p}}(c_p+c_v).$$

If M be the molecular weight of the gas and R the gas constant per gram molecule, then

$$p=\frac{RT}{M}\rho,$$

$$c_p-c_v=\frac{R}{M},$$

and therefore,

$$\epsilon_K=\sqrt{\frac{R}{8\pi}}\frac{1}{\sqrt{MT}}\frac{c_p+c_v}{c_p-c_v}.$$

The heat transfer was thus far expressed in mechanical units. To express it in calories, we have to take for ϵ_K the last value divided by E, the mechanical equivalent of a heat unit. With $E=419\cdot10^5$ and $R=83{\cdot}2\cdot10^6$ we find

$$\epsilon_K=43{\cdot}44\cdot10^{-6}\frac{1}{\sqrt{MT}}\frac{c_p+c_v}{c_p-c_v}.$$

With a somewhat different deduction of the formula Knudsen finds for the numerical factor the value $43{\cdot}46\cdot10^{-6}$.

We have thus found for the number of calories transferred per unit time and unit area from the plate S' to the plate S

$$W=(T'-T)p\epsilon_K.$$

The heat transfer appears then to be independent of the distance of the plates.

This formula can also be used to represent the heat transfer in other cases, provided the temperature difference $T'-T$ is small. Thus, for instance, we can imagine a hot body A (Fig. 10) kept at the temperature T' and surrounded by an enclosure B having a lower temperature T. Since all molecules impinging against A arrive from B (mutual collisions of molecules are excluded), they will have the energy ϵ corresponding to the temperature T, while the molecules coming from A have the energy ϵ' corresponding to T', the accommodation coefficient

being for the present left out of account. The heat supplied by A can thus be represented by

$$N(\epsilon' - \epsilon),$$

if N be the number of molecules received or sent out by A per unit area and unit time. The temperature difference being small, the distinction between the N-values corresponding to the temperatures T and T' is of no account, and we can take the number which would hold for $T' = T$. This number will be proportional to the pressure and independent of the shape and the dimensions of A and B. The factor $\epsilon' - \epsilon$ depends on the temperature difference $T' - T$ alone and is again independent of the shape of A and B. Whence it follows that the heat lost by A per unit area should be expressible by the formula

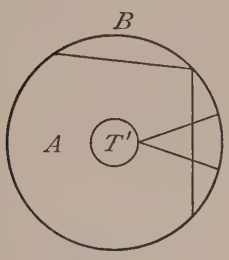

FIG. 10.

$$W = (T' - T)p\epsilon_K.$$

This formula, however, will not hold for the heat gained per unit area by B, since some molecules issuing from a certain part of the enclosing walls hit another part of these walls and not the body A. Only the total loss of heat of the whole surface of A will be equal to the gain of the whole surface of B.

20. THE EFFECT OF ACCOMMODATION UPON HEAT CONDUCTION

It appears from Knudsen's experimental findings that the amount of heat transferred from one to the other body and the coefficient ϵ_K calculated therefrom are smaller than their theoretical values. The heat conduction depends on the relative dimensions of the surfaces and on their nature. For rough surfaces the experimental and the theoretical values of ϵ_K differ from each other less than for smooth ones.

To account for this Knudsen considers *the effect of accommodation.*

Once more imagine the two plates S and S' (Fig. 8) at the temperatures T and T'. Let θ and θ' be the temperatures of the gas molecules leaving S and S', and a and a' the accommodation coefficients of these plates respectively.

According to the definition of the accommodation coefficient (cf. p. 134) we have the relations

$$\theta' - \theta = a(\theta' - T),$$
$$\theta - \theta' = a'(\theta - T'),$$

from which follows

$$\theta' - \theta = \frac{aa'}{1-(1-a)(1-a')}(T' - T).$$

In the expression (42) for the heat transfer $T' - T$ has now to be replaced by $\theta' - \theta$, which can be accomplished by writing

$$W = (T' - T)p\bar{\epsilon}_K,$$

where

$$\bar{\epsilon}_K = \frac{aa'}{1-(1-a)(1-a')}\epsilon_K.$$

Knudsen points out that what is measured in the experiments is this new coefficient $\bar{\epsilon}_K$. Among the special cases he considers first that in which both surfaces are of the same kind, so that $a = a'$, the coefficient being then denoted by ϵ_{11}, and then the case in which the plate S' is infinitely rough so that $a' = 1$ and the coefficient is denoted by $\epsilon_{1\infty}$. The coefficients ϵ_K, ϵ_{11}, and $\epsilon_{1\infty}$ are related to each other by the equations

$$\epsilon_{11} = \frac{a}{2-a}\epsilon_K, \quad \epsilon_{1\infty} = a\epsilon_K,$$

ϵ_K being the value of the coefficient when both plates are infinitely rough, so that it might also be written $\epsilon_{\infty\infty}$.

The magnitudes $\frac{1}{\epsilon_K}$, $\frac{1}{\epsilon_{1\infty}}$ and $\frac{1}{\epsilon_{11}}$ appear to form an arithmetical series.

Knudsen determined these different magnitudes by measurements in which the heat transfer took place between two coaxial cylinders. It appeared that the heat transfer depended here on the distance of the surfaces and was, for a given pressure, greater when the cylinders differed considerably in their radii than when they surrounded each other closely. This can be explained by noticing that with a large distance between the surfaces of the cylinders the molecules which leave the heated inner cylinder C' do not return to C' after a single collision with the outer cylinder C, as would be the case for a small distance

of the surfaces, but that between two collisions of a molecule with C' several collisions with C can take place, so that the molecule has more opportunities to give up heat to the inner side of the cylinder C. That the nature of the surface also has an effect upon the heat conduction will become plain by assuming that a colliding molecule gives up the more of its energy to a wall, the more rough the latter. In the case of a perfectly rough wall the molecules give up at a single collision their excess of heat, and the distance of the surfaces will, therefore, have no influence upon the rate of the heat conduction.

We will consider, then, the two co-axial cylinders C' and C, kept at the temperatures T' and T respectively, and in calculating the heat conduction we will follow Knudsen's not quite rigorous method. A perfectly rigorous treatment would offer too great difficulties.

Let us assume that a molecule which flies away from C' returns to C' after n collisions with C. Further, let the temperature of molecules on leaving C' be θ', after a single collision with C, θ_1, after two collisions with C, θ_2, and after n such collisions θ_n. If a is the accommodation coefficient for both surfaces, we have the relation

$$\theta' - \theta_1 = a(\theta' - T),$$

whence

$$\theta_1 - T = (1-a)(\theta' - T) = b(\theta' - T),$$

where $b = 1 - a$. After n collisions with C we have

$$\theta_n - T = b^n(\theta' - T),$$

and a collision with C' gives

$$\theta' - T' = b(\theta_n - T').$$

From these equations we derive

$$\theta' - \theta_n = a\,\frac{1-b^n}{1-b^{n+1}}\,(T' - T),$$

so that

$$\epsilon_K = a\,\frac{1-b_n}{1-b^{n+1}}\,\epsilon_K.$$

This expression gives us the amount of heat carried over from C' to C by molecules which between two collisions with C' collide n times with C.

Knudsen splits the heat transfer into parts contributed by molecules which between two collisions with C' collide with C once, or twice, and so on. For this purpose, however, one has first to find the probability that a particle which flies away from the outer cylinder C will hit the inner cylinder C'.

We have always assumed after Knudsen that the number of particles which are sent out by a surface-element in directions contained within a cone of aperture $d\omega$, and with an axis making an angle a with the normal to the surface-element, is proportional to $\cos a d\omega$. If we now consider a surface-element placed at a point P of the outer cylinder, the required probability will be represented by

$$w = \frac{\int_1 \cos a d\omega}{\int_2 \cos a d\omega},$$

the integral 1 being taken over all cones which have their vertex at the chosen point P of the outer cylinder and which intersect the inner cylinder, while the integral 2 has to be extended over all cones, with the same vertex P, within a solid angle 2π. We lay through P two planes (Fig. 11), PRA, passing through the common axis of the cylinders, and PRB, touching the inner cylinder. Let $QR = \phi$, $\angle QRA = \theta$, $\angle BRA = \bar{\theta}$. Then

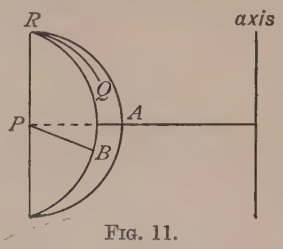

FIG. 11.

$$d\omega = \sin\phi d\theta d\phi.$$

Since a is the spherical distance QA, $\cos a = \sin\phi \cos\theta$, and therefore,

$$\omega = \frac{\int_0^{\bar{\theta}} \int_0^{\pi} \sin^2\phi \cos\theta d\theta d\phi}{\int_0^{\pi/2} \int_0^{\pi} \sin^2\phi \cos\theta d\theta d\phi} = \sin\bar{\theta} = \frac{r}{R},$$

where r is the radius of the cylinder C' and R that of C. Thus, of all molecules leaving C' a fraction r/R will again return to C' after a single collision with C. The remaining part $1 - r/R$ of the molecules will have collided with C more than once. Of these again the fraction r/R will return to C' after a second

collision with C. Thus, the part of all molecules which return to C' after two collisions with C is $\frac{r}{R}\left(1-\frac{r}{R}\right)$, and the part returning after n collisions, $\frac{r}{R}\left(1-\frac{r}{R}\right)^{n-1}$.

The total heat transfer can now be represented by

$$W=(T'-T)p\epsilon_{r,R},$$

where

$$\epsilon_{r,R}=\epsilon_K\sum_{n=1}^{n=\infty}a\frac{r}{R}\frac{1-(1-a)^n}{1-(1-a)^{n+1}}\left(1-\frac{r}{R}\right)^{n-1},$$

while

$$\epsilon_K=43{\cdot}46\,.\,10^{-6}\frac{c_p+c_v}{c_p-c_v}\frac{1}{\sqrt{MT}}. \quad . \quad . \quad (43)$$

This formula was tested by Knudsen in a series of experiments. He worked with two glass cylinders, of which the inner one consisted of a thin-walled tube around which many windings of platinum wire were coiled and fused into the glass. The radius of this tube was $r=0{\cdot}340$ cm. To the platinum coil two stouter platinum wires were attached by means of which the tube could be suspended within the wider glass cylinder. In the first measurements the radius of the outer cylinder was $R_1=0{\cdot}465$ cm., and in later ones $R_2=1{\cdot}61$ cm. The outer cylinder is placed in melting ice, while the inner one is heated by an electric current passing through the platinum coil. The resistance of the latter and the intensity of the electric current are measured and from these the temperature of the inner cylinder and the amount of heat generated are calculated. Since the measurements are made after the stationary state is established, this amount of generated heat is equal to that of transferred heat. The loss of heat was due partly to radiation and partly to molecular conduction. By means of a system of pipettes equal small amounts of gas could be introduced. This did not change the heat loss due to radiation but only that due to conduction, so that from a series of readings of the pressure, the temperature, and the amount of heat generated the amount of transferred heat per degree of temperature difference, per unit of pressure and unit of area, *i.e.* the coefficient $\epsilon_{r,R}$, could be computed. The gas chosen was hydrogen.

Knudsen found for ϵ_{r,R_1} $1{\cdot}87 \cdot 10^{-6}$ cal. and for ϵ_{r,R_2} $2{\cdot}45 \cdot 10^{-6}$ cal. These results show that, in accordance with the theory, the heat conduction increases with the distance of the cylindrical surfaces.

The ratio of ϵ_{r,R_1} and ϵ_{r,R_2} contains only the accommodation coefficient as unknown quantity, so that the latter can be computed from $\frac{\epsilon_{r,R_1}}{\epsilon_{r,R_2}}$. One finds $a = 0{\cdot}26$.

With this value of a the coefficient ϵ_K can be calculated. Its value thus found is $11{\cdot}1 \cdot 10^{-6}$, while the formula (43) based on the kinetic theory gives $\epsilon_K = 11{\cdot}0 \cdot 10^{-6}$,—a good agreement testifying to the correctness of the theory.

Some further measurements undertaken by Knudsen with the purpose of determining the change of the accommodation coefficient with the temperature need not detain us here.

With regard to the accommodation coefficient it may still be mentioned that from the theoretical standpoint it is not satisfactorily defined. In fact, the accommodation coefficient has a meaning only if the state of the gas is completely determined by the mean kinetic energy of the molecules, if, *e.g.*, it is assumed that Maxwell's law holds for the velocities of the molecules before as well as after their collision with a solid wall. This is for the gas between the two cylindrical surfaces not quite the case.

For a rigorous treatment of the question one would have to take into account, for a given temperature of the wall and a given velocity of the arriving molecules, the probability of a determined state of the molecule after it rebounds from the wall. This problem, however, is too intricate.

21. Heat Conduction in a Gas of Greater Density contained between Two Plates of Unequal Temperature

After this treatment of the heat conduction in strongly rarefied gases we will now give a short account of thermal conduction in a gas of greater density. Whereas it is known that the coefficient of heat conduction for large gas densities is independent of the density, the intermediate domain between the very small densities, explored by Knudsen, and great densities requires still a detailed investigation. We have already touched this domain when treating of a streaming gas

layer,* and we have then started from large densities. It appeared that with decreasing density next to friction also sliding began to assert itself. We will now inquire into the behaviour of the heat conduction coefficient in this domain.

We consider two plates kept at constant temperatures T and T'. Let the density of the gas be such that the mean free path of the molecules cannot be quite disregarded in comparison with the distance of the plates, although there are still very many collisions. The temperatures of the gas layers in contact with the plates will differ somewhat from T and T' as was already pointed out by Kundt and Warburg.

Fig. 12 gives a graphical representation of the temperatures of different layers. OO' is the distance of the plates, OA and $O'A'$ represent the temperatures T and T' of the plates, and OB and $O'B'$ those of the adjoining gas layers as measured by the mean kinetic energy of the molecules. For large densities the temperature of the gas layers may be represented by the straight line AA'.

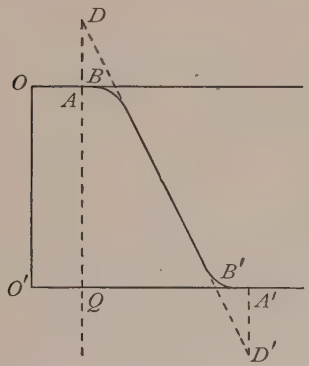

FIG. 12.

At some distance from the fixed plates the theory for large densities holds, so that the temperature gradient is uniform and the temperature line has there a straight portion. We produce the latter up to the intersection points D and D' with the vertical lines through A and A'. We assume that $T' - T$ is infinitesimal, in case the heat conduction coefficient should depend on temperature. This condition enables us to assume that if the temperatures of the plates are increased or diminished by equal amounts, the line does not change its shape.

The whole figure can thus be shifted horizontally without changing the length of the lines AD and $A'D'$. Again, the figure can be transformed by increasing or diminishing all the distances from AQ in the same ratio. In this case also the lengths of AD and $A'D'$ remain unchanged; they are thus independent of the temperatures of the plates. If both plates are of

* Cf. Chapter II. Art. 10.

the same kind, we should find the same curve by interchanging their temperatures. This curve will be found by taking the mirror image of the original one with respect to AQ. Whence it follows that the parts of the curve at B and B' are congruent, and that therefore $AD = A'D'$.

Moreover, since the curve retains its shape when the distance of the plates is changed,* the length $AD = A'D'$ for a given gas will depend only upon the nature of the plates and will be independent of their temperature and their distance.

Let the distance of the plates be l and $AD = A'D' = \Delta$.

Similarly to what was said on p. 104 about the significance of OC and BD in Fig. 2, a physical meaning can be ascribed to Δ. In fact, if the gas extended also outside of the plates and had throughout the same temperature gradient as actually exists within the gas, the temperatures at the points D and D' would be equal to those of the plates, *i.e.* T and T' respectively.

The temperature gradient in the gas at not too small distances from the plates is then $\frac{T'-T}{l+2\Delta}$ and the heat transfer is $k\,\frac{T'-T}{l+2\Delta}$, where k denotes the conduction coefficient.

Whence it follows that if the plate distance l is increased the amount of heat transferred does not vary inversely proportionally to l but slower.

We will now prove that, for given temperatures of the plates, Δ is inversely proportional to the density while k is independent of the latter.

For this purpose we compare two geometrically similar cases [systems] in one of which the gas density is n times greater and the dimensions n times smaller than in the other. Let P_1 and P_2 be two corresponding points, and let the state at these points, apart from the density, be exactly the same. Let ds be a volume element at P_1 and $Fdsd\lambda$ the number of molecules contained in it, whose velocity components and quantities determining the inner state fall within a given domain $d\lambda$. A similar group in an equal element at P_2 will then contain $\frac{1}{n}\,Fdsd\lambda$ molecules. We

* That $AD = A'D'$ is independent of the distance of the plates can be seen by inserting between these a third plate P of such a temperature as not to change the state of the gas apart from the immediate neighbourhood of P, and by applying a similar reasoning to that used on p. 103 to prove that the sliding coefficient ν is independent of the distance of the plates.

assume that the temperatures of the plates are the same in both cases; then the temperatures of gas layers at corresponding points will also be equal. This harmonises with the assumption that at corresponding points the number of the molecules in different groups in one gas is the same fraction of their number in the other gas; therefore the averages of all quantities are equal, and this holds then also for the temperature as the mean energy.

We will first of all prove that the possibility of the existence of the first state carries with it also that of the second. We consider in the first state a group of particles and follow their history during a certain time τ. The parameters characterising these particles are contained within a given domain, while the particles themselves lie in a volume element ds. If there were no collisions, we would find these particles at the end of the time τ in a volume element ds'. Owing to the collisions some particles will leave the group and others will join it. The condition that the state should be stationary implies that there are in ds' at the beginning and at the end of the time τ equal numbers of particles whose velocities and parameters determining the inner state fall within certain limits. We assume such to be the case for the first state and shall prove that this holds then also for the second state.

The quantities concerning the first state will be distinguished by the suffix 1 and those relating to the second state by the suffix 2.

We consider in both cases a group of particles with equal intervals for the magnitudes characterising them and contained within equal volume elements $(ds)_1 = (ds)_2$ placed at corresponding points. We take the time interval τ for the second system n times as long. Then also $(ds')_1$ and $(ds')_2$ will lie at corresponding points. If collisions are disregarded, the number of particles contained in $(ds')_1$ at the beginning of that time interval will be n times that contained in $(ds')_2$. This then will also be the case at the end of the time intervals τ_1 and τ_2, since all these particles arrive from $(ds)_1$ and $(ds)_2$. If there are collisions, then things are not so simple. It may be noticed, however, that the number of particles which leave the group owing to collisions and the number of those that join it for the same reason amount in both systems to the same fraction of the total number of particles. In order to see this, the collisions may be classified according to circum-

stances. Under coinciding circumstances the number of particles in the first system is n times, and, therefore, the number of collisions n^2 times as large as in the second system. But since the time interval for the second system was taken n times as large, the number of particles leaving or joining the group in the first system will not be n^2 but only n times as large. The ratio of densities will thus remain in both systems the same; also with collisions will $(ds')_1$ contain n times as many molecules as $(ds')_2$.

For the interior of the gas it is thus proved that if the state of the first system is stationary, so is that of the second. And if we further assume that the particles rebound from both plates in the same way, this theorem will hold also for the limiting gas layers at the plates.

Lastly, to prove that Δ is inversely proportional to the density, we note that the graphical representation of the temperature distribution for the second case can be simply deduced from that of the first. In fact, since we have assumed that the respective temperatures of the plates are equal in both cases and that this is true also of the temperatures of gas layers at corresponding points, the horizontal dimensions of the two figures are equal, while the vertical ones of the second are n times those of the first. Whence it follows that $\Delta_2 = n\Delta_1$, which proves that Δ is inversely proportional to the density of the gas. It remains only to prove that the conduction coefficient k is independent of the density.

22. Heat Conduction Coefficient Independent of Density

In order to prove this, we consider the number of molecules which pass in equal times through unit area of corresponding planes V_1 and V_2 parallel to the plates, and we compare again similar groups of molecules.

This number is for the first case n times as great as that for the second, and this holds then also for the total energy carried across the surface-element. Consequently, the heat conduction in the first case is n times that of the second, and since the temperature gradient is also n times as great, the coefficient k will be the same for both cases.

Moreover, noticing that Δ and k do not change when at constant density of the gas the plates are moved farther apart or brought nearer to each other, we can say, generally, that Δ is inversely proportional to the density and that k is independent of density.

23. Lasareff's Experimental Investigation

Lasareff * investigated the temperature distribution in the immediate neighbourhood of a wall. He took a very thin gas layer, about 9 mm. thick, and worked with highly rarefied gases. As walls, metal plates were used which were kept at constant temperatures by means of water jackets, a cold jacket being placed under the lower and a warm one above the upper plate. The temperatures were measured by means of a thermo-electric pile of which one junction was placed near the cold plate and the other could be moved up and down in the space between the plates. Fig. 13 gives a graphic representation of Lasareff's experimental results. The abscissae represent the temperature difference relatively to the cold plate and the ordinates the distance from the heated plate. The curves *AB*, *CD*, *EF*, and *GH* give the observed results at a pressure of 760, 0·087, 0·065, and 0·019 mm. mercury respectively.

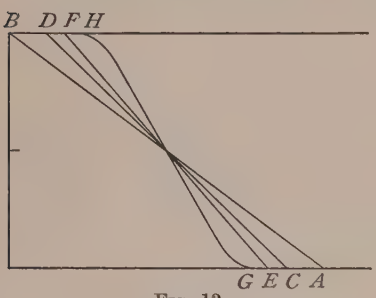

Fig. 13.

Lasareff found for hydrogen :

p (mm. mercury)	γ
4·5	0·022
2	0·055

γ is a coefficient, proportional to Δ. Inasmuch as it can be assumed that the temperature of the wires of the thermo-pile actually coincides with that of the gas, these experimental results prove satisfactorily that Δ is inversely proportional to the density.

* Lasareff, *Ann. der Phys.*, xxxvii., 1912, p. 233.

CHAPTER IV

REMARKS ON LESAGE'S THEORY OF GRAVITATION

24. Lesage's Theory of Gravitation

Knudsen's investigations on rarefied gases may be connected with the old theory of gravity due to Lesage.* According to this theory celestial space is full of small particles or corpuscles moving in all directions with great velocities. Material bodies are incessantly hit by these corpuscles and throw them back. A

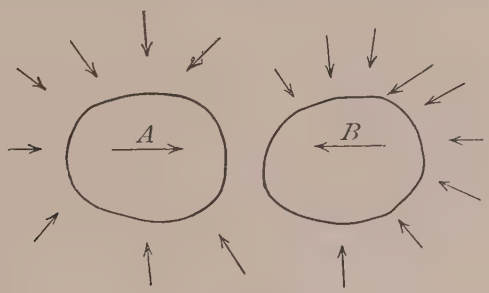

Fig. 14.

body placed alone in space will be hit in all directions by equal numbers of particles and will thus experience no resultant effect. Two bodies A and B, however, would partly shield each other from the impact of the corpuscles and consequently appear to exert upon each other an attractive force (Fig. 14). If the corpuscles are assumed to be small compared with the atoms, this would lead to an attractive force between two atoms.

Since the number of corpuscles intercepted by one atom is proportional to the solid angle under which it is seen from the

* *Journal des savants*, 1764.

other atom, the force would be inversely proportional to the square of distance of the two atoms.

But it does not follow from the theory that the force is proportional to the masses; it would, instead, depend on the dimensions of the bodies. To secure the proportionality of the force to mass, yet another hypothesis would have to be introduced.

Meanwhile the theory of Lesage was shown by Maxwell to be incorrect. The motion of the corpuscles is much the same as that of gas particles, and as against the fact that the body B intercepts corpuscles which in its absence would have reached the body A there is this other fact, that due to reflection from B some corpuscles will reach A which otherwise would not do so.

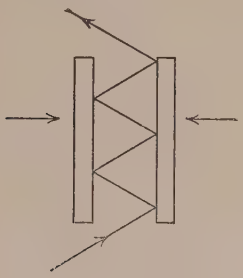

Fig. 15.

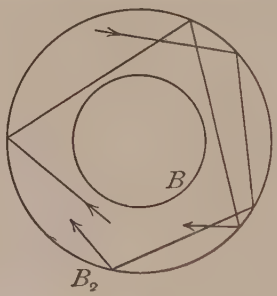

Fig. 16.

It is not possible to keep the space between two bodies free from corpuscles, no more than to keep a space free from black body radiation, even if the mean free path of the corpuscles is great and their mutual collisions may be disregarded. In the case of two parallel plane plates, for instance, only those corpuscles which hit the plates perpendicularly would not reach the space between the plates; but this is only an infinitesimal fraction of the total number of corpuscles.

It would be difficult to prove that a space can never remain free from corpuscles. In fact, one can very well imagine a particular state of motion for which a certain domain remains free from corpuscles, if, *e.g.*, these move originally outside a certain sphere B and inside a second sphere B_2 concentric with B. If the surface of B_2 is perfectly reflecting and if there are no collisions between the corpuscles, then the corpuscles will never penetrate into the inner sphere (Fig. 16). But such a state of motion will never arise.

The theory of Lesage can be saved by assuming that the corpuscles are wholly or partially absorbed by matter. But then the picture is deprived of its simplicity.

25. An Electromagnetic Analogue of Lesage's Theory

It was once asked whether Lesage's theory can be given an electromagnetic form. One would then have to assume, for instance, that space is full of radiation of a wave-length much shorter than that of Röntgen-rays and to show that two particles would be driven towards each other by the radiation pressure. Such would, in fact, be the case if the particles continually absorbed the radiation.

Consider two electrically charged particles P and Q, at a mutual distance r, in a space traversed by electromagnetic waves. Let E be the electric, H the magnetic force, and n the frequency. Evidently the particle P will be set into vibrations, whether it is free or bound to a position of equilibrium; only the type of vibrations will be different in the two cases. Owing to the radiation emitted from Q the radiation field in which P is placed will be modified.

Let us introduce a system of co-ordinates with P as origin, and let the co-ordinates of Q be $-r$, 0, 0.

In a first approximation the displacement components of P can be written

$$x = aeE_x - be\dot{E}_x,$$
$$y = aeE_y - be\dot{E}_y,$$
$$z = aeE_z - be\dot{E}_z.$$

This is a convenient way of expressing the phase difference between the incident waves and the oscillation produced by them. The phase difference is represented by the second terms and is caused by the resistance experienced by the particle due to friction or emission of radiation. The coefficients a and b are supposed to be constants which depend on the mass of P, on the restituting force, etc. In E_x, E_y, E_z is included the field which the particle Q, being also set vibrating, produces around itself by its radiation.

Noticing that the work done by the electric force against the

resistance, *i.e.* $eE_x\dot{x}+eE_y\dot{y}+eE_z\dot{z}$, for a full period, is positive, it will be seen that b must be positive.

As soon as P is set vibrating, the force upon it is no longer E as at the origin [position of equilibrium]. The x-component of the force can now be written, in a second approximation,

$$\frac{e}{c}(\dot{y}H_z-\dot{z}H_y)+e\left(x\frac{\partial E_x}{\partial x}+y\frac{\partial E_x}{\partial y}+z\frac{\partial E_x}{\partial z}\right)+eE_x,$$

where for H_y, H_z, E_x and their derivatives are to be taken the values belonging to the original, *i.e.* the equilibrium position of P. The first term is due to the velocity of the particle, and the second due to its deviation x, y, z from the position of equilibrium. These terms give rise to very weak oscillations with twice the frequency of the incident waves.

Now, to find the x-component of the force acting upon P, the last expression has to be averaged over a full period. The required, rather lengthy calculations, in which account must also be taken of the disturbance of the field by the vibrations of Q, may here be omitted. It appears that, if terms with $1/r^3$, etc., be neglected (which is permitted provided r is large compared with the wave-length), the term $\frac{e}{c}(\dot{y}H_z-\dot{z}H_y)$ alone survives, and this also inasmuch only as it depends on the terms $-be\dot{E}$ in the preceding equations.

As a result of these calculations one finds for the required force

$$-b\frac{e^2}{c}(\ddot{E}_yH_z-\ddot{E}_zH_y)=\frac{b^2n^2e^2}{c}(E_yH_z-E_zH_y),$$

where for E and H is to be taken the field as it is in presence of Q, but neglecting the effect of P upon it.

This result can be associated with an energy flux. In fact, the expression $c(E_yH_z-E_zH_y)$ represents the x-component of the energy flux at P. Since we assume that the rays are propagated equally in all directions, there will be in all points of a sphere with Q as centre and of radius r the same radial energy flux. If E be the flux of energy through the whole spherical surface, reckoned positive when directed outwards, then

$$c(E_yH_z-E_zH_y)=\frac{\mathrm{E}}{4\pi r^2}.$$

The component of the force upon P taken along the X-axis will thus be represented by

$$\frac{bn^2e^2}{c^2}\,\frac{\mathrm{E}}{4\pi r^2}.$$

Since Q is placed on the negative X-axis, this expression will represent an attractive force if it is negative, that is to say, if E is negative. Such will be the case if more energy streams through the sphere inwards than outwards, and therefore, if Q absorbs the rays.

The electromagnetic modification of Lesage's theory leads thus to a similar result as the original corpuscular theory.

CHAPTER V

FRICTION AND HEAT CONDUCTION IN THE PROPAGATION OF SOUND

26. The Effect of Friction

We will now consider the effect of inner friction and of heat conduction upon the propagation of sound in a gas.

Let u, v, w be the velocity components of a volume element and s the relative condensation, *i.e.* if ρ be the actual density and ρ_0 that in the original state of equilibrium, $s = (\rho - \rho_0)/\rho_0$. Further, let $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = K$, and let us for the present disregard the heat conduction. Then the equations of motion for the propagation of sound waves will be

$$K + \frac{\partial s}{\partial t} = 0, \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$\left.\begin{aligned} a^2 \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} - \nu\left(\Delta u + \frac{1}{3}\frac{\partial K}{\partial x}\right) &= 0 \\ a^2 \frac{\partial s}{\partial y} + \frac{\partial v}{\partial t} - \nu\left(\Delta v + \frac{1}{3}\frac{\partial K}{\partial y}\right) &= 0 \\ a^2 \frac{\partial s}{\partial z} + \frac{\partial w}{\partial t} - \nu\left(\Delta w + \frac{1}{3}\frac{\partial K}{\partial z}\right) &= 0 \end{aligned}\right\} . \quad . \quad . \quad (45)$$

Here a would be the velocity of propagation of sound in absence of inner friction, and $\nu = \mu/\rho$, where μ is the friction coefficient or the viscosity.*

* Formula (44) is the equation of continuity for a compressible fluid, viz.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0,$$

in which the terms $u\frac{\partial \rho}{\partial x}$, etc., are neglected, the deviations from equilibrium being assumed infinitesimal, and the term $\frac{1}{\rho}\frac{\partial \rho}{\partial t}$ is replaced by $\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} = \frac{\partial s}{\partial t}$.

Differentiating the three equations (45) with respect to x, y, z and taking account of (44) we have an equation for s,

$$a^2\Delta s - \frac{\partial^2 s}{\partial t^2} + \frac{4}{3}\nu\frac{\partial \Delta s}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad (46)$$

This is, for $\nu = 0$, the well-known equation of propagation of sound waves.

The problem of propagation in an unlimited gas mass can be easily solved for the case of plane waves.

The equations (45) follow from the equations of motion

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \text{ etc.} \quad . \quad . \quad (a)$$

(cf. p. 77).

The expressions for the stress components can be found by following the hints given in the footnote to page 78, with the only difference that our fluid is now compressible so that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = K$ does not vanish.

Thus one finds

$$X_x = -p + (a_{11} - a_{22})\frac{\partial u}{\partial x} + a_{22}K,$$

$$Y_z = \frac{a_{11} - a_{22}}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

and similar formulae with the same coefficients a_{11} and a_{22} for the remaining stress components. In these equations p is the pressure, as it would correspond to the density and temperature at the given point if the gas were at rest or if it had throughout the same velocity u, v, w.

Between a_{11} and a_{22} holds, moreover, the relation $(a_{11} - a_{22}) + 3a_{22} = 0$, which can be deduced by going deeper into the manner of arising of the stresses, but which can be accepted here without proof.

Ultimately, putting $a_{11} - a_{22} = 2\mu$, we find

$$X_x = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu K,$$

$$Y_z = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

and substituting these expressions for the stress components in (a),

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\Delta u + \frac{1}{3}\mu\frac{\partial K}{\partial x}, \text{ etc.} \quad . \quad . \quad (b)$$

Assuming Poisson's law $p = c\rho^\kappa$, we have

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} = \frac{\kappa p}{\rho}\frac{\partial s}{\partial x}.$$

Again, $\frac{\kappa p}{\rho} = a^2$, while $\frac{1}{\rho_0}\frac{\partial p}{\partial x}$ can be replaced by $\frac{1}{\rho}\frac{\partial p}{\partial x}$, so that

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = a^2\frac{\partial s}{\partial x}.$$

This, substituted in (b), gives the equations (45), if we put $\mu/\rho = \nu$ and neglect products of velocities into their derivatives.

If we put $s = s_0 e^{int - qx}$, we can find at once for every frequency the corresponding value of q and thence the propagation velocity and the damping of sound.

We will consider here the propagation in a gas contained in a cylindrical tube, whose axis coincides with the x-axis. The propagation is to proceed along the x-axis, so that all terms will contain the factor $e^{int - qx}$.

Since $\frac{\partial s}{\partial t} = ins$ and $\frac{\partial^2 s}{\partial t^2} = -n^2 s$, equation (46) becomes

$$(a^2 + \tfrac{4}{3}\nu in)\Delta s + n^2 s = 0. \quad . \quad . \quad . \quad . \quad (47)$$

Since the state in a cylindrical tube is symmetrical about its axis, we introduce, instead of y and z, the distance r from the axis, so that

$$\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

Again, $\frac{\partial^2 s}{\partial x^2} = q^2 s$, and therefore, instead of (47),

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \left(\frac{n^2}{a^2 + \frac{4}{3}i\nu n} + q^2\right)s = 0.$$

This is the differential equation of the Bessel function I_0, so that we have, for the condensation,

$$s = c_1 e^{int - qx} I_0(Ar), \quad . \quad . \quad . \quad . \quad (48)$$

where

$$A^2 = \frac{n^2}{(a^2 + \frac{4}{3}i\nu n)} + q^2.$$

The argument of I_0 is thus a complex number.

The value of u is now to be found from the first of equations (45), where $K = -\frac{\partial s}{\partial t} = -ins$, so that this equation reduces to

$$\frac{\partial u}{\partial t} - \nu\Delta u = q(a^2 + \tfrac{1}{3}i\nu n)s. \quad . \quad . \quad . \quad . \quad (49)$$

Of this a particular solution can be found by putting $u = \xi s$ and by suitably choosing the constant ξ. Substituting this in the left-hand member of (49), we have

$$\frac{\partial u}{\partial t} - \nu\Delta u = \xi\left(\frac{\partial s}{\partial t} - \nu\Delta s\right) = \xi(ins - \nu\Delta s),$$

so that

$$\xi(ins - \nu\Delta s) = q(a^2 + \tfrac{1}{3}i\nu n)s. \quad . \quad . \quad . \quad (50)$$

Since, by (47),
$$\Delta s = -\frac{n^2}{a^2 + \frac{4}{3}i\nu n}\, s,$$
we see that equation (50) is satisfied by
$$\xi = q\left(-i\frac{a^2}{n} + \tfrac{4}{3}\nu\right).$$

To this particular solution for u an arbitrary solution of the equation $\frac{\partial u}{\partial t} - \nu\Delta u = 0$ can still be added.

Since u is a function of x and r only, the last equation can be written
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \left(q^2 - \frac{in}{\nu}\right)u = 0,$$
and this is again satisfied by a Bessel function,
$$u = c_2 e^{int - qx} I_0(Br), \text{ where } B^2 = q^2 - \frac{in}{\nu}.$$
Thus the solution for u becomes
$$u = e^{int - qx}[\xi c_1 I_0(Ar) + c_2 I_0(Br)]. \quad . \quad . \quad . \quad (51)$$

It remains only to determine the velocity components v and w. Owing to the symmetry the velocity in a cross-section of the tube will be radial, so that v and w can be represented by $v = yh$ and $w = zh$, where h will, apart from the factor $e^{(int - qx)}$, depend on r alone. If these values be substituted in the second and the third of equations (45), all terms in the second equation will have the factor y/r and all terms in the third the factor z/r. On being divided by y and z respectively, these equations turn out to be identical and after a multiplication by r each of them reduces to
$$(a^3 + \tfrac{1}{3}in\nu)\frac{\partial s}{\partial r} + (in - \nu q^2)rh - \nu\left(3\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right) = 0. \quad . \quad (52)$$

The third term of (52) can be evaluated by means of the equation of continuity (44) which can be written
$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -ins + qu.$$

Substituting here $v=yh$ and $w=zh$ and differentiating with respect to r, we have

$$3\frac{\partial h}{\partial r}+r\frac{\partial^2 h}{\partial r^2}=-in\frac{\partial s}{\partial r}+q\frac{\partial u}{\partial r},$$

and this introduced into (52) gives

$$-(in-\nu q^2)rh=(a^2+\tfrac{4}{3}in\nu)\frac{\partial s}{\partial r}-q\nu\frac{\partial u}{\partial r}.$$

Since $\frac{\partial s}{\partial r}$ and $\frac{\partial u}{\partial r}$ follow from the solutions (48) and (51) for s and u, the radial velocity rh is herewith determined.

To determine the constants c_1 and c_2, we have two boundary conditions. For greater densities, when there is no sliding, the gas is at rest at the walls, so that for $r=R$ (radius of the cross-section of the tube), $u=0$ and $rh=0$. This gives the equations

$$\xi c_1 I_0(AR)+c_2 I_0(BR)=0,$$
$$(a^2+\tfrac{4}{3}in\nu-q\nu\xi)c_1 A I_0'(AR)-q\nu c_2 B I_0'(BR)=0,$$

where I_0' is written for the derivative of I_0.

Eliminating from these two equations c_1/c_2, we find

$$(a^2+\tfrac{4}{3}in\nu-q\nu\xi)A\frac{I_0'(AR)}{I_0(AR)}+q\nu\xi B\frac{I_0'(BR)}{I_0(BR)}=0.$$

Here q is the only unknown. Since q is contained also in A and B, its determination from this equation is very laborious. One has to use approximation formulae for the Bessel function and to assume that the effect of friction is small and thus also that the state differs but little from the propagation in a non-viscous gas.

27. Effect of Heat Conduction

This problem was treated by Kirchhoff, who has taken into account also the influence of the heat conduction.

In deducing the equations of motion (45) use was made of Poisson's law for adiabatic volume changes (cf. footnote on p. 157). If, however, heat conduction is taken into account, Poisson's law can no longer be applied, and one has to write down the thermical equation which concerns the change of the internal energy of a volume element due to compression and heat conduction.

Kirchhoff* finds for the propagation velocity in a cylindrical tube of diameter $2R$

$$v = a\left(1 - \frac{\gamma}{2R\sqrt{\pi n}}\right), \quad . \quad . \quad . \quad . \quad (53)$$

where n is the number of oscillations per second and a the propagation velocity in an unlimited three-dimensional space, in which case the effect of friction is small; γ is a constant depending on viscosity and heat conduction, viz.

$$\gamma = \sqrt{\nu} + \left(\frac{a}{b} - \frac{b}{a}\right)\sqrt{k},$$

where

$$\nu = \frac{\text{viscosity}}{\text{density}},$$

$$k = \frac{\text{coefficient of heat conduction}}{\text{density}},$$

while b is the propagation velocity as calculated by Newton, so that $\frac{a}{b} = \sqrt{\frac{c_p}{c_v}}$.

Formula (53) was repeatedly tested experimentally but was never found well corroborated. Kayser† found the deviation of the propagation velocity from the value a about four times as large as that required by the theory.

In a space of three dimensions everything becomes much simpler than in a tube. The effect of viscosity is then the greater the smaller the wave-length.

Neklepajev‡ investigated, in connection with Lebedew's theoretical researches,§ the propagation of very short sound waves in air. His source of sound was an electric spark produced at the focus of a concave mirror S_1. The beam of parallel rays (for such short waves, as *e.g.* 0·2 cm., one can speak of "sound rays") reflected by the mirror S_1 fell upon a diffraction grating consisting of a series of silvered steel rods. Diffracted bundles were thus produced, and one such bundle was concentrated by a second concave mirror S_2 upon a sensitive vane which was

* *Pogg. Ann.*, cxxxiv., 1868, p. 177; *Ges. Abh.*, Leipzig, 1882, p. 540.
† *Wied. Ann.*, ii., 1877, p. 218.
‡ *Ann. der Phys.*, xxxv., 1911, p. 175.
§ *Ibid.*, xxxv., 1911, p. 171.

displaced by the pressure of the sound rays. The wave-lengths were measured by means of the grating, through the diffraction angle. Neklepajev worked with wave-lengths of 2·5 down to 0·85 mm.

By sending the sound rays through layers of air of different thickness the absorption could be measured. This appeared to be considerable. The experiments gave a higher value for the absorption than was to be expected from Lebedew's theoretical considerations. Lebedew gives for the distance in which the intensity is reduced to $\frac{1}{100}$ of its original value, for different wave-lengths, the following figures :

λ in mm.	Distance in cm.
0·8	40
0·4	10
0·2	2·5
0·1	0·6

For polyatomic gases, in addition to ν and k yet a third coefficient must be introduced. In fact, when the temperature rises while a volume element is being compressed, this will increase the velocity of the translational motion of the molecules as well as the intensity of their inner motion. In the state of equilibrium the energy of translational motion bears a determined ratio to that of the inner motion, but while the velocity of translation is directly affected by the compression of a volume element, the effect upon the velocity of the inner motion is not so immediate. The internal energy remains thus in its fluctuations, so to speak, behind the energy of translation. This gives a coefficient affecting the propagation velocity of sound, viz. making it somewhat smaller. It remains to be seen, however, whether this coefficient can have a perceptible value.

CHAPTER VI

KINETIC THEORY OF SYSTEMS OF ELECTRONS RICHARDSON'S INVESTIGATIONS

28. Theoretical Introduction

Richardson* has made an important investigation on the emission of negative electrons by a hot metal, and found that this is due to the heat motion. The electrons, endowed with great velocities, will escape in spite of the forces exerted by the metal. The escaping electrons were found by Richardson to have a kinetic energy agreeing with that of gas molecules.

We will, first of all, follow here Maxwell's considerations on the velocity distribution in a monatomic gas acted upon by an external force, as *e.g.* the gravity.

Let ξ, η, ζ be the velocity components and x, y, z the co-ordinates of a gas molecule. We consider the molecules whose velocity components and co-ordinates are contained between the limits ξ and $\xi+d\xi$, η and $\eta+d\eta$, ζ and $\zeta+d\zeta$, x and $x+dx$, y and $y+dy$, z and $z+dz$. Their number will be represented by

$$fd\xi d\eta d\zeta dxdydz = fds.$$

The components of the external force will be denoted by X, Y, and Z.

All this being valid for the instant t, let us now follow the history of this particular group of particles. Mutual collisions being excluded, these particles will be contained at the instant $t+dt$ within a phase element ds' placed at a point of the six-dimensional space whose co-ordinates are

$$\xi+\frac{X}{m}dt,\ \eta+\frac{Y}{m}dt,\ \zeta+\frac{Z}{m}dt,\ x+\xi dt,\ y+\eta dt,\ z+\zeta dt.$$

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* *Phil. Mag.* (6), xvi., 1908, p. 353.

Let the state of the gas be stationary. This implies that the number of particles which fall within the phase element ds' is the same at the instant t as at the instant $t+dt$, so that

$$f\left(\xi+\frac{X}{m}dt,\ \ldots,\ x+\xi dt,\ \ldots\right)ds'=f\,ds.$$

But by Liouville's theorem $ds'=ds$, so that

$$f\left(\xi+\frac{X}{m}dt,\eta+\frac{Y}{m}dt,\zeta+\frac{Z}{m}dt,x+\xi dt,y+\eta dt,z+\zeta dt\right)=f(\xi,\eta,\zeta,x,y,z)$$

or, since the difference of these functions is zero,

$$\frac{\partial f}{\partial\xi}\frac{X}{m}+\frac{\partial f}{\partial\eta}\frac{Y}{m}+\frac{\partial f}{\partial\zeta}\frac{Z}{m}+\frac{\partial f}{\partial x}\xi+\frac{\partial f}{\partial y}\eta+\frac{\partial f}{\partial z}\zeta=0.$$

This equation can be satisfied by

$$f=ae^{-h(\xi^2+\eta^2+\zeta^2)},$$

where a is a function of the co-ordinates and h is independent of them.

In fact, on substituting we find (with $v^2=\xi^2+\eta^2+\zeta^2$)

$$-2\frac{h}{m}a(\xi X+\eta Y+\zeta Z)+\xi\left(\frac{\partial a}{\partial x}-av^2\frac{\partial h}{\partial x}\right)$$

$$+\eta\left(\frac{\partial a}{\partial y}-av^2\frac{\partial h}{\partial y}\right)+\zeta\left(\frac{\partial a}{\partial z}-av^2\frac{\partial h}{\partial z}\right)=0,$$

an equation which must be satisfied for all values of ξ, η, ζ, x, y, z, so that the coefficients of ξ, ξ^3, etc., must all vanish. This gives

$$\frac{\partial h}{\partial x}=0,\ \frac{\partial h}{\partial y}=0,\ \frac{\partial h}{\partial z}=0,\ \frac{\partial a}{\partial x}=\frac{2ha}{m}X,\ \frac{\partial a}{\partial y}=\frac{2ha}{m}Y,\ \frac{\partial a}{\partial z}=\frac{2ha}{m}Z.$$

Thus, the external force must have a potential. If χ be the corresponding potential energy,

$$X=-\frac{\partial\chi}{\partial x},\ Y=-\frac{\partial\chi}{\partial y},\ Z=-\frac{\partial\chi}{\partial z},$$

and

$$a=a_0e^{-\frac{2h}{m}\chi}.$$

Since $f = ae^{-h(\xi^2+\eta^2+\zeta^2)}$, we find in the well-known way $3/2h$ for the mean square of the velocity and $a\left(\frac{\pi}{h}\right)^{3/2}$ for N, the number of particles per unit volume.

This can be written $N = N_0 e^{-\frac{2h}{m}\chi}$, if N_0 be the number of particles at such places at which the potential of the external force is nil. This shows that the density of particles is smallest where the potential energy is greatest.

Since in what precedes no assumption was made about the peculiarities of the field of force, apart from the existence of a potential, the results arrived at may be utilised in our further considerations on the motion of electrons.

We begin with a limiting case in which the external force acts only within a thin layer and is directed normally to the boundary of the two media. Let χ_1 be the potential energy in the first, and χ_2 that in the second medium. Both magnitudes are constant, while the potential energy changes discontinuously across the boundary. If the medium 1 be a metal, and the medium 2 the space above it, then the velocity distribution of the particles in both media obeys Maxwell's formula, the value of h being the same for both.

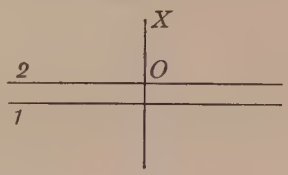

FIG. 17.

Thus the densities will be given by

$$a_1 = a_0 e^{-\frac{2h}{m}\chi_1}, \quad a_2 = a_0 e^{-\frac{2h}{m}\chi_2}.$$

Owing to the force in the boundary layer the velocity of the particles will be diminished in traversing this layer. In spite of this the mean velocity of the particles will be the same in both media, the reason being that only those particles leave the first medium which have the greatest velocity.

To test this result by an explicit treatment, let us introduce a co-ordinate system of which the yz-plane coincides with the boundary of the two media and the positive x-axis extends into the second medium.

We consider a particle [electron] which at the instant t is contained in the medium 1 and whose co-ordinates, $x, y, z, \xi, \eta, \zeta$ fall within the phase element ds. Let $x', y', z', \xi', \eta', \zeta'$ be its

co-ordinates at an instant $t+\tau$, where τ is a finite time interval, long enough for the particle to have crossed the boundary at the instant $t+\tau$. It will be assumed that it does not collide with atoms or with other electrons.

The condition that the sum of the potential and the kinetic energy should be constant gives the equations

$$\left.\begin{aligned} \xi'^2 &= \xi^2 - \frac{2}{m}(\chi_2 - \chi_1), \\ \eta' &= \eta, \\ \zeta' &= \zeta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (54)$$

Again, noticing that the time required to reach the boundary is $-x/\xi$, we find the relations

$$\left.\begin{aligned} x' &= \xi'\left(\tau + \frac{x}{\xi}\right), \\ y' &= y + \tau\eta, \\ z' &= z + \tau\zeta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55)$$

With the aid of (54) and (55) it can be easily verified that the state is stationary. In fact, at the instant t there are fds particles within the phase element ds and these will have passed at the instant $t+\tau$ into the phase element ds', while the latter contained $f'ds'$ particles at the instant t. The condition for a stationary state will thus be

$$fds = f'ds'.$$

Now it can be readily shown that $ds = ds'$ and $f = f'$. In fact,

$$\frac{ds'}{ds} = \begin{vmatrix} \frac{\partial\xi'}{\partial\xi} & 0 & 0 & 0 & 0 & 0 \\ . & \frac{\partial\eta'}{\partial\eta} & 0 & 0 & 0 & 0 \\ . & . & \frac{\partial\zeta'}{\partial\zeta} & 0 & 0 & 0 \\ . & . & . & \frac{\partial x'}{\partial x} & 0 & 0 \\ . & . & . & . & \frac{\partial y'}{\partial y} & 0 \\ . & . & . & . & . & \frac{\partial z'}{\partial z} \end{vmatrix}$$

$$= \frac{\partial\xi'}{\partial\xi}\frac{\partial\eta'}{\partial\eta}\frac{\partial\zeta'}{\partial\zeta}\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}\frac{\partial z'}{\partial z} = \frac{\xi}{\xi'}\frac{\xi'}{\xi} = 1.$$

Again, $$f = a_1 e^{-h(\xi^2+\eta^2+\zeta^2)}, \quad f' = a_2 e^{-h(\xi'^2+\eta'^2+\zeta'^2)},$$

and by (54), taking account of the values of a_1 and a_2, these two expressions will be seen to be equal to each other.

We now ask how many electrons pass, in the state under consideration, from 1 to 2. We determine first the number of particles which, per unit time, pass through a surface-element $d\sigma$ normal to the x-axis in the first medium, and whose velocity components are contained between ξ and $\xi + d\xi$, η and $\eta + d\eta$, ζ and $\zeta + d\zeta$. For this number we find $\xi f d\xi d\eta d\zeta d\sigma$. Similarly, $\xi' f' d\xi' d\eta' d\zeta' d\sigma'$ will be the number of particles which pass, per unit time, through a surface-element $d\sigma'$ normal to the x-axis in the second medium, and whose velocity components fall within the limits ξ' and $\xi' + d\xi'$, η' and $\eta' + d\eta'$, ζ' and $\zeta' + d\zeta'$.

Let us now place $d\sigma$ and $d\sigma'$ in the immediate neighbourhood of the boundary and assume that ξ', η', ζ' belong to ξ, η, ζ, *i.e.* that ξ', η', ζ' are the velocities, after traversing the boundary, of a particle which in the first medium had the velocities ξ, η, ζ.

It follows then from what precedes that

$$\xi' d\xi' = \xi d\xi, \quad \eta' = \eta, \quad \zeta' = \zeta, \text{ and } f' = f.$$

If, therefore, $d\sigma = d\sigma'$, then also

$$\xi f d\xi d\eta d\zeta d\sigma = \xi' f' d\xi' d\eta' d\zeta' d\sigma'.$$

Herewith is also given the number of electrons which escape from the metal. If the state of the first medium is stationary, the number of electrons leaving it, per unit time, will be constant, even if the state of the second medium is not stationary, which is *e.g.* the case when this medium is unlimited so that the electrons are not thrown back.

At the beginning of the heating the metal plate sends out also positive ions, which emission ceases, however, after a certain time. These ions come probably from a layer of gas condensed at the surface of the plate.

29. Validity of Maxwell's Distribution Law for the Free Electrons in a Metal

The ratio e/m for the negative particles emitted by the plate was determined by J. J. Thomson,* who was able to ascertain

* *Phil. Mag.* (5), xlviii., 1899, p. 547.

that its value was the same as that known to belong to the electrons. Thus we have here the same corpuscles. Richardson measured their velocity, the arrangement of his experiment being as follows.

Into a quadrangular aperture cut out in a platinum plate P (Fig. 18) was fitted the protruding part of the bent platinum strip S, the latter being insulated from the plate P by mica.

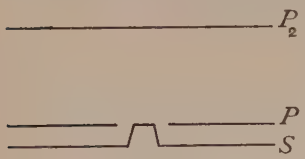

Fig. 18.

The strip S was heated by an electric current. Another metal plate P_2, connected with an electrometer, formed with the plate P a condenser. The plate P_2 gathered the electrons escaping from the heated strip S.

Richardson measured the potential of the plate P_2 during the process of charging, taking at the same time the utmost care that the centre of the platinum strip, the glowing spot from which the electrons were emitted, should remain at the potential zero.

The velocity with which the potential of the upper plate mounts in its negative value will gradually decrease, because owing to the electric field thus produced not all the electrons leaving the strip S can reach the upper plate (Fig. 19).

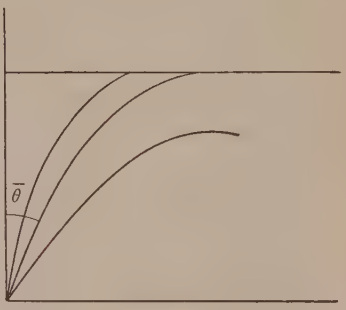

Fig. 19.

Let us first assume that all electrons leave the strip with the same velocity u. There is then a maximum value Φ_m to the potential of the upper plate. This will be attained when the electrons which move normally to the plates are just able to reach the upper plate. This gives the condition

$$\tfrac{1}{2}mu^2 = e\Phi_m,$$

whence

$$\Phi_m = \frac{mu^2}{2e}.$$

We can also find how the potential of the upper plate gradually increases. We will assume that the electrons which at the

potential Φ just reach the plate have left the heated strip at an inclination $\bar{\theta}$ to the normal. These electrons describe then a parabola whose apex lies at the upper plate (Fig. 19), and we have, for Φ,

$$\tfrac{1}{2}mu^2 \cos^2 \bar{\theta} = e\Phi. \quad . \quad . \quad . \quad . \quad (56)$$

Let now n be the total number of electrons which, per unit time, leave the heated strip, and therefore $2n \sin\theta \cos\theta d\theta$ the number of those among them which move in a direction contained between θ and $\theta + d\theta$ (cf. p. 128). Thus, the number of electrons for which θ is smaller than $\bar{\theta}$ is

$$2n\int_0^{\bar{\theta}} \sin\theta \cos\theta d\theta,$$

or, in virtue of (56),

$$n\left(1 - \frac{2e\Phi}{mu^2}\right).$$

For $\Phi = 0$ this becomes equal to n, which means that then all electrons reach the upper plate.

If C be the capacity of the condenser, the charge acquired by the plate P_2, per unit time, is

$$C\frac{d\Phi}{dt} = en\left(1 - \frac{2e\Phi}{mu^2}\right),$$

whence,

$$\Phi = \frac{mu^2}{2e}\left(1 - e^{-\frac{2ne^2}{mCu^2}t}\right).$$

For $t = \infty$ this gives Φ_m, while for $t = 0$ we have assumed $\Phi = 0$.

The intensity of the current is

$$i = C\frac{d\Phi}{dt} = nee^{-\frac{2ne^2}{mCu^2}t}.$$

Thus, from the measured value of Φ_m we can determine $mu^2/2e$, and from the rate of increase of the potential the value of $2ne^2/mCu^2$. From these two magnitudes ne can be calculated and, since e/m is known, also the velocity u can be determined.

We have thus far assumed that all the corpuscles leave the hot strip with the same velocity. Let us now see how the above result is modified if the velocities are distributed according to Maxwell's formula. It will appear that under these circumstances the potential of the upper plate does not attain a

maximum. In fact, when the potential of the plate has risen so high that the corpuscles of mean velocity do not reach it any more, corpuscles endowed with a much higher velocity will reach the plate and charge it to a yet higher potential.

Let now the upper plate be a circular disc, and let us assume the field between the plates to be homogeneous, the potential difference of the plates being Φ. We have to calculate how many of the corpuscles leaving an element of the heated strip placed at a point O will reach an arbitrary element of the upper plate. In doing so we shall assume the heated element to be very small compared with the receiving plate, so that all electrons which can overcome the potential difference will also be picked up by that plate. Let a be the distance of the two plates. If ξ_0, η, ζ are the components of the initial velocity of an electron, the number of electrons which, per unit time, leave the lower plate with velocity components contained between ξ_0 and $\xi_0 + d\xi_0$, η and $\eta + d\eta$, ζ and $\zeta + d\zeta$, will be represented by

$$a\xi_0 e^{-h(\xi_0^2+\eta^2+\zeta^2)}\, d\xi_0 d\eta d\zeta.$$

The coefficient a will be determined by integrating over ξ_0 from 0 to ∞ and over η and ζ from $-\infty$ to $+\infty$ and equating the integral to n, the total number of corpuscles, which, per unit time, leave the element of the hot strip. This gives

$$a = \frac{2}{\pi} h^2 n.$$

The integrated equation of motion of a corpuscle which left the glowing element ($x = 0$) at the instant $t = 0$ with a velocity ξ_0 in the direction of the X-axis is

$$x = \xi_0 t - \frac{e\Phi}{2ma} t^2.$$

This gives, for $x = a$, the time t required by the corpuscle to reach the upper plate. If the corpuscle actually reaches the plate, this equation has two real roots, the smaller of which is the required value of t.

We solve for $1/t$, and have therefore to take the greater of the two roots. Thus,

$$\frac{1}{t} = \frac{\xi_0}{2a} + \sqrt{\frac{\xi_0^2}{4a^2} - \frac{e\Phi}{2ma^2}}.$$

This expression will be denoted by g.

Let now a rectangular element of the upper plate be hit whose sides are dy, dz and which is placed at a point y, z. Since $y=\eta t$, $z=\zeta t$ or $\eta=gy$ and $\zeta=gz$, the element will be reached provided that η lies between gy and $g(y+dy)$, and ζ between gz and $g(z+dz)$.

The number of corpuscles which leave the heated strip with the X-component of velocity contained between ξ_0 and $\xi_0+d\xi_0$ and reach the given surface-element of the upper plate will thus be

$$a\xi_0 e^{-h\xi_0^2-hg^2(y^2+z^2)}g^2 d\xi_0 dy dz,$$

and the total number ν of corpuscles hitting that element will be found by integrating this expression over the values of ξ_0 for which g is real. Thus,

$$\nu = a dy dz \int_{\sqrt{2e\Phi/m}}^{\infty} \xi_0 e^{-h\xi_0^2-hg^2(y^2+z^2)} g^2 d\xi_0. \quad . \quad . \quad (57)$$

This, integrated for y and z over the upper plate, gives the required number of electrons as a function of Φ. Multiplying this number by the charge e of an electron we shall have the current charging the upper plate, that is, $Cd\Phi/dt$. Richardson calls this *the thermionic current.*

In Richardson's experiments the second [upper] plate was so large as compared with the mutual distance of the two plates that it may be considered as infinitely extended. This facilitates the integration of (57). Inverting the order of integrations we find

$$a\int_s^{\infty} g^2\xi_0 e^{-h\xi_0^2} d\xi_0 \int_{-\infty}^{+\infty} e^{-hg^2y^2} dy \int_{-\infty}^{+\infty} e^{-hg^2z^2} dz$$

$$= -\frac{a\pi}{h}\frac{1}{2h}\left| e^{-h\xi_0^2} \right|_s^{\infty} = \frac{a\pi}{2h^2} e^{-\frac{2e\Phi h}{m}} = ne^{-\frac{2e\Phi h}{m}}, \quad . \quad (58)$$

where $s=\sqrt{2e\Phi/m}$ and n is the total number of electrons leaving, per unit time, the hot strip.

The differential equation for the potential of the upper plate thus becomes

$$C\frac{d\Phi}{dt} = ene^{-\frac{2e\Phi h}{m}} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Its solution is

$$e^{\frac{2eh}{m}\Phi} = 1 + 2\frac{e^2hn}{mC}t, \quad . \quad . \quad . \quad . \quad (60)$$

where we have assumed $\Phi=0$ for $t=0$.

According to theory Φ should thus mount with t continually, though exceedingly slowly, the potential tending to become logarithmically infinite. Due to unavoidable leakages, however, Φ will practically reach a maximum after some finite time.

For the thermionic current we find ultimately, by (59) and (60),

$$i = C\frac{d\Phi}{dt} = \frac{en}{1+\frac{2e^2hn}{mC}t},$$

so that the current decreases continually, tending to zero for $t=\infty$.

These theoretical results were tested experimentally by Richardson. The value of en is determined from the measured intensity of the current at the beginning of the experiment and that of $\frac{2e^2hn}{mC}$ from the time rate of change of the current, and from these two magnitudes the value of eh/m can be calculated.

Now, $h=3/2u^2$, where u^2 is the mean squared velocity of the electrons, within as well as outside the metal. Whence,

$$\frac{eh}{m} = \frac{3e}{2mu^2}.$$

It is now assumed that the mean energy of an electron is equal to that of a gas molecule, so that $\frac{1}{2}mu^2 = \frac{3}{2}kT$, and therefore,

$$\frac{eh}{m} = \frac{e}{2kT}.$$

Again, if N be the number of molecules per gram molecule,

$$\frac{eh}{m} = \frac{eN}{2kNT} = -\frac{E}{2RT},$$

where R is the gas constant per gram molecule, and E (positive) the charge of a gram ion of a monovalent electrolyte. The value of E is thus known from electrolysis.

Now, having derived eh/m from his experiments and measured the temperature T of the hot metal, Richardson deduced by means of the last-written formula the value of the gas constant, for a quantity of gas, however, which at the temperature 0° C. and the pressure of 76 cm. occupies a volume of 1 cm.^{3}.

Since the value of R per gram molecule (2 grams of hydrogen) is $83{\cdot}2 \cdot 10^6$ and since 1 cm.^{3} of hydrogen at 0° and 76 cm. weighs

0·0000898 gram, Richardson's theoretical value of the gas constant, which will be denoted by $\bar{R}$, should be 3730.

The results of the experiments, in which the wire [platinum strip] had temperatures from 1473° to 1813° and was heated during 16 up to 35 hours, were as follows :

$$\begin{aligned} R = & 4{\cdot}1 \,.\, 10^3 \\ & 4{\cdot}2 \,.\, 10^3 \\ & 3{\cdot}5 \,.\, 10^3 \\ & 3{\cdot}6 \,.\, 10^3 \\ & 2{\cdot}9 \,.\, 10^3 \\ & 3{\cdot}1 \,.\, 10^3 \\ & 3{\cdot}2 \,.\, 10^3 \\ & 3{\cdot}4 \,.\, 10^3 \end{aligned}$$

In a later experiment $4{\cdot}04 \,.\, 10^3$ was found.

The satisfactory outcome of these experiments proves that the free electrons in a metal have the same mean kinetic energy as a gas molecule and that for their velocity distribution Maxwell's law is valid.

30. Velocity Distribution of Thermions for Different Directions

In the experiments described in Art. 29 it depends only upon the velocity component perpendicular to the heated plate whether an electron reaches the plate P_2 or not. The agreement with Maxwell's law is thus actually proved only for the component ξ. In a second series of experiments * Richardson investigated the distribution of the velocity components η and ζ. The arrangement of these experiments was such that the charging of the upper plate was mainly due to corpuscles which left the hot plate in a slanting direction.

The electrons are here emitted from a long narrow metal strip R (Fig. 20) which fills out almost completely a narrow slit in the lower plate P_1. This strip is placed perpendicularly to the plane of the drawing. The plates are very extended. The upper plate P_2 has a narrow slit BC parallel to that at R. The electrons flying through this slit enter into the metal cylinder E which is insulated from the plate. The plate P_2 and the cylinder can be

* *Phil. Mag.* (6), xvi., 1908, p. 890 ; (6), xviii., 1909, p. 681.

alternately connected with an electrometer and both can be shifted together in a horizontal direction. Thus it can be determined in each position what part of the total thermionic current passes through the slit BC.

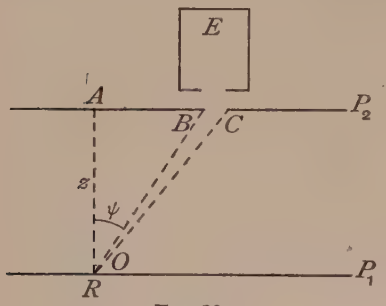

Fig. 20.

The measurements are made while the electric field is yet very weak.

Let us now assume that Maxwell's law holds for all the velocity components and that for the emission of the electrons there is the same distribution over different directions as for gas molecules flying through a surface-element. In order to calculate the number of particles which, starting from an element $d\sigma$ placed at a point O (Fig. 20) of the hot metal strip, pass through the slit BC of the upper plate, we construct around O a sphere with $OA = z$, the distance of O from the upper plate, as radius. Two planes laid through O and the edges of the slit BC will cut this sphere along two great circles (Fig. 21). Let $AB = x$, $\angle AOB = \psi$, and $\angle YOP = \theta$. The electrons flying across $d\sigma$ with a velocity u in the direction OP fill out a cylinder with $d\sigma$ as base and the normal velocity component $u \cos(AP)$ as height. Their number is thus proportional to $\cos(AP)$. If we now count up the electrons flying through $d\sigma$ with different velocities but all in the direction OP, the result will still be proportional to $\cos(AP)$, and if this direction falls within a narrow cone of aperture $\sin\theta d\theta d\psi$, the number of electrons will also be proportional to the latter. Since, in the triangle APE, $\cos(AP) = \sin\theta \cos\psi$, that number can be written

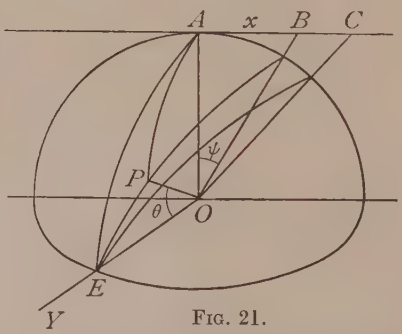

Fig. 21.

$$c \cos\psi \sin^2\theta \, d\theta d\psi,$$

where c is a constant. Thus the total number of electrons sent towards the slit will be

$$c \cos\psi \, d\psi \int_0^\pi \sin^2\theta d\theta = \tfrac{1}{2}\pi c \cos\psi d\psi.$$

Now, $\tan\psi = x/z$, and therefore, $\dfrac{d\psi}{\cos^2\psi} = \dfrac{dx}{z}$.

The required number can therefore be written

$$\tfrac{1}{2}\pi c \frac{dx}{z}\left(\frac{z^2}{z^2+x^2}\right)^{3/2}.$$

Richardson's measurements are in good agreement with this result. The number of electrons hitting a narrow strip of the upper plate attains a maximum for $x=0$, while for $x=\infty$ it tends to zero.

It is thus proved that for the electrons emitted by a hot metal the velocity distribution is the same for all directions ; otherwise c would not be constant.

31. The Work required to drive an Electron out of the Metal

Richardson * undertook also some further investigations with the object of measuring the work required to drive an electron out of the metal plate. For this purpose he investigated how the emission of the electrons depends on temperature, having assumed in this connection the number of corpuscles per unit volume of the metal to be independent of temperature.

Let us consider a surface-element of unit area placed in the second medium near at and parallel to the boundary of 1 and 2. In order to determine the number n of corpuscles passing through this element per unit time, we introduce again the function f (cf. pp. 164 and 165), viz. for the metal

$$f_1 = a_0 e^{-\frac{2h}{m}\chi_1} e^{-hv^2},$$

and for the second medium

$$f_2 = a_0 e^{-\frac{2h}{m}\chi_2} e^{-hv^2}.$$

---

* *Phil. Trans.* A, cci., 1903, p. 497.

We consider first the corpuscles whose velocities are contained between v and $v+dv$ and directions of motion between θ and $\theta+d\theta$, where θ is the angle between the direction of motion and the normal of the boundary surface.

The number of these corpuscles passing per unit time across the said surface-element is

$$a_0 e^{-\frac{2h}{m}\chi_2} e^{-hv^2} 2\pi v^3 \sin\theta \cos\theta d\theta dv. \quad . \quad . \quad (61)$$

Integrating this expression over θ and v we find for the total number of corpuscles crossing that surface-element, per unit time,

$$n = 2\pi a_0 e^{-\frac{2h}{m}\chi_2} \int_0^{\pi/2} \sin\theta \cos\theta d\theta \int_0^{\infty} v^3 e^{-hv^2} dv = \frac{\pi a_0}{2h^2} e^{-\frac{2h}{m}\chi_2}. \quad (62)$$

If S_1 be the number of corpuscles per unit volume of the metal, we have

$$S_1 = 4\pi \int_0^{\infty} a_0 e^{-\frac{2h}{m}\chi_1} v^2 e^{-hv^2} dv = a_0 e^{-\frac{2h}{m}\chi_1} \sqrt{\frac{\pi^3}{h^3}}, \quad . \quad (63)$$

whence

$$n = \frac{1}{2\sqrt{\pi h}} e^{-\frac{2h}{m}(\chi_2 - \chi_1)} S_1. \quad . \quad . \quad . \quad (64)$$

Now, $h = \dfrac{3}{2u^2}$, so that

$$\frac{2h}{m} = \frac{3}{mu^2} = \frac{1}{kT}. \quad . \quad . \quad . \quad . \quad (65)$$

We introduce further the quantity $\psi = -\chi/e$, which will be the potential energy for a unit of negative charge, and we write

$$-\frac{e}{kT} = -\frac{eN}{RT} = \frac{E}{RT}, \quad . \quad . \quad . \quad (65')$$

where E is the (positive) charge of a gramion of a monovalent electrolyte and R the gas constant.

The formula for n then becomes

$$n = \tfrac{1}{2}\sqrt{\frac{2}{3\pi}}\, u e^{-\frac{E}{RT}(\psi_2 - \psi_1)} S_1.$$

Here u is proportional to $\sqrt{T}$.

By means of this formula, n being observed as a function of temperature, the difference $\psi_2 - \psi_1$, which is positive, can be determined.

Richardson found from his measurements $\psi_2 - \psi_1 = 4{\cdot}1$ volts. Investigations by Wilson * and by Deininger,† and a second determination by Richardson,‡ gave a somewhat greater value. The mean of these determinations, 5·54 volts, agrees well with the value 5·5 volts found by Richardson § in a later investigation.

32. The Density of Electrons in the Metal

From Richardson's experimental findings we may also derive the number S_1 which is here considered as independent of temperature. For this purpose we transform the formula for n by making use of the relation

$$\tfrac{1}{2}mu^2 = \tfrac{3}{2}kT,$$

whence follows

$$u = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3}{eN}\frac{e}{m}kNT} = \sqrt{-\frac{3e}{m}\frac{RT}{E}}.$$

The expression under the radical is positive, since e stands for the negative charge of the electron. Thus we have

$$n = \tfrac{1}{2}\sqrt{-\frac{2T}{\pi}\frac{R}{E}\frac{e}{m}}\,\mathrm{e}^{-\frac{E}{RT}(\psi_2-\psi_1)}S_1.$$

From the value of n that of S_1 can now be calculated, taking $\psi_2 - \psi_1 = 5{\cdot}5$ volts. Richardson, however, does not give the value of n itself, so that this has to be calculated from other magnitudes measured by him. The experimental procedure was as described on p. 174 (cf. Fig. 20). The slit and the cylinder E were placed at A above the middle of the heated strip R. The cylinder E picked then up all the electrons hitting an area of the breadth of the slit BC and of the length of 1 cm. The upper plate, the slit, and the cylinder could now be shifted in a direction perpendicular to the slit BC and thus also perpendicular to the heated strip R. In each of the successive positions electrons were collected by the cylinder during 30 seconds. The electrons which hit the cylinder during a displacement over the full breadth of the plate are those which would reach a strip β, 1 cm. broad,

* *Phil. Trans.* A, ccii., 1903, p. 243.
† *Ann. der Phys.*, xxv., 1908, p. 285.
‡ *Phil. Trans.* A, ccvii., 1906, p. 1.
§ *Phil. Mag.* (6), xx., 1910, p. 205.

of the upper plate within 30 seconds; the hot strip R is perpendicular to the strip β.

From Richardson's measurements the charge carried over, per unit of time, from R to β is found to be $1{\cdot}63 \,.\, 10^{-9}$ coulombs. And since the charge of an electron amounts to $1{\cdot}5 \,.\, 10^{-19}$ coulombs, this gives for the number of electrons emitted by R per second and hitting β, 10^{10}. In order to calculate from this number the number n of electrons emitted by the metal strip per 1 cm.^{2} and per second, we note that the number of electrons flying from R towards β is equal to that sent out by a rectangle of area $R\beta$ cm.^{2}, if R and β be the breadth in cms. of the strips denoted by these letters. In Richardson's experiments, as mentioned before, β was equal to 1 cm., while the breadth of R amounted in some experiments to 0·02 cm., and in some others to 0·04 cm. Thus, taking $R = 0{\cdot}03$ cm., we find $n = 10^{10}/0{\cdot}03 = 33 \,.\, 10^{10}$.

Again,

$$\begin{aligned} \psi_2 - \psi_1 &= 5{\cdot}5 \,.\, 10^8 \text{ electromagnetic units,} \\ E &= 9650 \quad \text{,,} \quad \text{,,} \\ R &= 83{\cdot}2 \,.\, 10^6 \quad \text{,,} \quad \text{,,} \\ -e/m &= 1{\cdot}77 \,.\, 10^7 \quad \text{,,} \quad \text{,,} \\ T &= 1050 + 273 = 1323. \end{aligned}$$

This gives for the number of electrons per unit volume of the metal $S_1 = 10^{25{\cdot}7}$.

The result is not very satisfactory, the number found being much too large. This is due to the high value of $\frac{E}{RT}(\psi_2 - \psi_1)$ which we will denote by W. In fact, we have found

$$n = \tfrac{1}{2}\sqrt{\frac{2}{3\pi}}\, u e^{-\frac{E}{RT}(\psi_2 - \psi_1)} S_1 = \tfrac{1}{2}\sqrt{\frac{2}{3\pi}}\, u e^{-W} S_1.$$

Now, in order that the electrons might escape freely we should have $\psi_2 = \psi_1$, and therefore,

$$n = \text{const. } u S_1.$$

Owing to the factor e^{-W} the number of electrons escaping per unit area and unit time will be much smaller than what by the last formula would correspond to the number (S_1) per unit volume of the metal and will decrease considerably with decreasing temperature. The latter is in accordance with facts.

In Richardson's experiments $W = 48$, so that $S_1 = 2n\sqrt{\frac{3\pi}{2}} \cdot \frac{1}{u} e^{48}$ [*i.e.* $10^{25{\cdot}7}$].

Such a large number of free electrons per unit volume, however, is hard to accept. This is even much greater than the number of platinum atoms per cm.^{3}, for which one finds $8 \cdot 10^{22}$.* Moreover, if it be granted that the electrons take part in the heat motion and each of them has the mean kinetic energy $\frac{3}{2}kT$, such a large number of free electrons would give a much too high value for the specific heat of the metal.

For the number of corpuscles in the second medium we find, per unit volume,

$$S_2 = 4\pi \int_0^\infty a_0 e^{-\frac{2h}{m}\chi_2 v^2} e^{-hv^2} dv = a_0 e^{-\frac{2h\chi_2}{m}} \sqrt{\frac{\pi^3}{h^3}},$$

or by (62),

$$S_2 = 2\sqrt{\pi} hn = 2\sqrt{\frac{3\pi}{2}} \frac{1}{u} n = 2\sqrt{\frac{\pi m}{2kT}}\, n.$$

On the other hand, S_1 can be written

$$S_1 = 2\sqrt{\frac{\pi m}{2kT}}\, e^{\frac{E}{RT}(\psi_2 - \psi_1)} n, \quad . \quad . \quad . \quad (66)$$

so that the density of electrons in the metal should be $e^{\frac{E}{RT}(\psi_2 - \psi_1)}$ times that in the second medium, as we already know.

33. Difference in Potential Energy between the Electrons inside and outside the Metal Atoms

The conclusion of Art. 32 can be avoided by assuming that the electrons are partly bound to the atoms and partly free. In addition to the force in the boundary layer which hinders the electrons from escaping freely from the metal, there is then still another force which binds some of the electrons within the metal to the atoms. We assume that the potential energy in the metal within an atom has a definite value, and we denote by ψ_1 the potential energy per unit of negative charge for an electron outside, and by ψ_1' that for an electron within the atom, with the understanding that ψ_1' is smaller than ψ_1. Further, we denote the part of the space occupied by the atoms by a, and therefore the remaining part by $1 - a$.

* In fact, 1 gram atom of Pt weighs 195 gr.; 1 cm.^{3} of Pt weighs 21·5 gr., and thus amounts to $\frac{1}{9}$ gram atom, while the number of atoms in a gram atom can be put at $68 \cdot 10^{22}$.

For the number of electrons per unit volume of the metal we thus find

$$S_1 = 2\sqrt{\frac{\pi m}{2kT}}\left[(1-\alpha)e^{-\frac{E}{RT}\psi_1} + \alpha e^{-\frac{E}{RT}\psi_1'}\right]e^{\frac{E}{RT}\psi_2}n, \quad . \quad (67)$$

where we have assumed that the medium 1 is in equilibrium with the medium 2 and that the density varies with the ψ's as indicated by the expression $e^{\frac{E}{RT}(\psi_2-\psi_1)}$. Our task is now to represent the observed facts by means of this formula. Assuming S_1 constant, we have to find how n depends upon T, to compare this dependence with the experimental results, and thence to calculate ψ_1 and ψ_1'. With these exponents we should then be able to deduce again S_1 from n.

Two extreme cases may be distinguished. The first, corresponding to $\psi_1' = \psi_1$, has just been considered. In this case the electrons were subjected only to forces within the boundary layer.

The second limiting case is that the electrons are acted upon by forces only within the atoms and that there are no boundary forces. In this case ψ_1 will tend to ψ_2 and $\psi_2 - \psi_1'$ will be much greater than $\psi_2 - \psi_1$, so that, even if α be small, the first term can be omitted in a first approximation. This gives for S_1 the same formula as in the previous reasoning, but with the factor α, and with ψ_1 replaced by ψ_1'. From the experiments it follows then again that $\psi_2 - \psi_1' = 5{\cdot}5$ volts. This, however, gives now for S_1 a value which is only the small fraction α of that found before, as if the volume were α times smaller and as if the electrons within it were very strongly bound. By choosing α small enough the proper value for S_1 can be obtained.

One might object to the formula (67) on account of its being based upon the assumption of a large number of electrons in each atom. That the formula is at any rate correct * can be shown by a method of reasoning due to Gibbs.†

In fact, the number of systems in a canonic assemblage, with co-ordinates contained between q_1 and $q_1 + dq_1$, q_2 and $q_2 + dq_2$, . . . q_n and $q_n + dq_n$, is

$$e^{-E_q/\Theta} dq_1 dq_2 \ldots dq_n, \quad . \quad . \quad . \quad (68)$$

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* See, however, the remark at the end of this article.

† *Elementary Principles in Statistical Mechanics*, New York, London, 1902.

where E_q stands for the potential energy which corresponds to the given conditions and Θ is proportional to the temperature. If all mutual action between the parts is disregarded, the potential energy of the system is equal to the sum of the potential energies of its parts and the expression for the number of systems is split into factors, so that (68) assumes the form

$$e^{-E_1/\Theta} dx_1 dy_1 dz_1 \; e^{-E_2/\Theta} dx_2 dy_2 dz_2 \ldots .$$

In our case x_1, y_1, z_1 will be the co-ordinates of a corpuscle and E_1 its potential energy.

The probability that a corpuscle is situated at a given place is thus independent of the remaining corpuscles. The ratio of the number of systems in which the first corpuscle is within the part α of the space to the number of systems in which it is outside that part of the space is then

$$\alpha e^{-E_i/\Theta} : (1-\alpha) e^{-E_u/\Theta}, \qquad . \quad . \quad . \quad (69)$$

where E_i is the potential energy of the corpuscle within and E_u that outside the part α of the space. The same expression gives for each system the ratio of the number of corpuscles contained within the part α of the space to the number of corpuscles lying outside, and thus formula (67) is established.

In this deduction no assumption whatever was made with regard to the number of electrons contained within the part α of the space, so that it could even be applied to the case in which there are fewer electrons than atoms. What was assumed, however, and what may seem objectionable, is that statistical mechanics can be applied to the electrons within the atoms.

CHAPTER VII

VACUUM CONTACT OF PLATES OF DIFFERENT METALS

34. Potential Difference for Vacuum Contact

In connection with Richardson's investigations the following considerations present themselves.

Suppose we had two metal plates, both at the same constant high temperature, and placed in an exhausted space at a certain distance from each other (vacuum contact). If the temperature is high enough, electrons will be emitted by either plate. If the two plates are of different metals, the number of these electrons will not be the same for both, and this will produce a potential difference of a determined equilibrium value, which must be equal to the potential difference between the metals when placed in direct contact. For, if such were not the case, the two plates could be connected by a wire and, although everything is kept at the same constant temperature, we should have an electric current which would clash with the second law of thermodynamics.

Let the plates be placed horizontally and let the lower plate A have a higher potential than the upper, B, the potential difference being Φ.

By (58), the number of electrons leaving, per second, the lower and reaching the upper plate at a potential difference Φ (which diminishes the velocity of the electrons and thus prevents some of them from reaching the upper plate) is

$$ne^{-\frac{2e\Phi h}{m}} = ne^{-\frac{E\Phi}{RT}},*$$

* Cf. p. 176, (65) and (65′). In (58) Φ was the potential of the *upper* plate with respect to the lower one.

where n is the total number of electrons emitted, per second, by the lower plate. In view of (66), n can be written

$$\tfrac{1}{2}\sqrt{\frac{2kT}{\pi m}}\, e^{\frac{E}{RT}\psi_a} S_a,$$

where ψ_a (the previous $\psi_1 - \psi_2$) is the difference in potential energy per unit of negative charge within and without the metal A, and S_a the number of electrons per unit volume of this metal.

This gives for the number of electrons flying, per second, from A to B

$$\tfrac{1}{2}\sqrt{\frac{2kT}{\pi m}}\, e^{\frac{E}{RT}(\psi_a - \Phi)} S_a,$$

while the number of those flying, per second, from B to A is

$$\tfrac{1}{2}\sqrt{\frac{2kT}{\pi m}}\, e^{\frac{E}{RT}\psi_b} S_b,$$

the factor $e^{-\frac{E}{RT}\Phi}$ being here left out, since the electric field does not counteract the motion of these electrons, so that all of them reach the plate A.

In the case of equilibrium both numbers must be equal, hence the equation

$$e^{\frac{E}{RT}(\psi_a - \psi_b - \Phi)} = \frac{S_b}{S_a}.$$

This gives for Φ

$$\frac{E}{RT}\Phi = \frac{E}{RT}(\psi_a - \psi_b) - \log \frac{S_b}{S_a}.$$

The ratio S_b/S_a is thus in the present case also given by an exponential law in which appears the potential energy, split into an electrostatic part and one which depends on the difference in the attraction between the metals and the electrons.

The same law for the ratio S_b/S_a holds also in the case of a direct contact between the plates A and B, and since S_a and S_b have fixed values, the potential difference in the latter case is the same as for plates separated by a vacuum.

We can introduce here the same modifications as in the preceding investigation by dividing the electrons within the metal into two kinds, the intra-atomic and the extra-atomic ones.

Let the potential energy of an electron *in vacuo* be by $-ep$

greater than the potential energy of an electron in the metal outside the atoms, and by $-ep'$ greater than that of an electron within a metal atom. Since e is always the negative value of the charge of an electron, both of these expressions are positive. Let, further, $E/RT = \mu$.

Then, by formula (67),

$$n = \tfrac{1}{2}\sqrt{\frac{2kT}{\pi m}}\,\frac{1}{(1-\alpha)e^{\mu p} + \alpha e^{\mu p'}}\,S.$$

For the case of the two plates A and B under consideration we find, in view of the potential difference Φ (the plate A having the higher potential),

$$n_1 e^{-\mu\Phi} = n_2,$$

$$\mu\Phi = \log\frac{n_1}{n_2},$$

$$\mu\Phi = \log\frac{S_1}{S_2}\cdot\frac{(1-\alpha_2)e^{\mu p_2} + \alpha_2 e^{\mu p_2'}}{(1-\alpha_1)e^{\mu p_1} + \alpha_1 e^{\mu p_1'}},$$

the suffixes 1 and 2 being attached to all magnitudes belonging to the plates A and B respectively.

In the case of a direct contact between the metals the same formula will hold for the equilibrium value of the potential difference.

35. Resistance at a Vacuum Junction

Let us now consider a thermo-electric element with two vacuum junctions which are kept at the temperatures T and T' (Fig. 22).

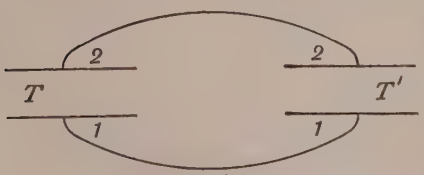

Fig. 22.

This will produce a current, the electromotive force being the resultant of two potential differences at the junctions while the element is open, and of possible potential differences in the metals due to the temperature difference.

Now, we may ask whether this current has to overcome also a resistance at the vacuum junctions. To answer this question

we assume that the potential difference Φ between the plates differs very little from the equilibrium potential difference Φ_0.

From the preceding considerations it follows that the number of electrons passing from the plate 1 (of higher potential) to the plate 2 is $n_1 e^{-\mu\Phi}$. The number moving in the opposite direction is n_2, so that the electric current (reckoned positive from 1 to 2) will be

$$i = e(n_1 e^{-\mu\Phi} - n_2).$$

Again,

$$0 = e(n_1 e^{-\mu\Phi_0} - n_2),$$

so that

$$i = en_1(e^{-\mu\Phi} - e^{-\mu\Phi_0}).$$

If $\mu\,(\Phi - \Phi_0)$ is small, we have approximately

$$i = -e\mu n_1 e^{-\mu\Phi_0}(\Phi - \Phi_0),$$

so that there is, in a certain sense, a resistance at the vacuum junctions which is equal to

$$-\frac{1}{e\mu n_1} e^{\mu\Phi_0} = -\frac{1}{e\mu n_2}.$$

This is positive, since e is negative.

36. Peltier Effect for a Vacuum Contact

To close this subject, let us still consider the question of the Peltier effect for the case of such a vacuum contact. If the effect were here the same as in the case of metal plates in direct contact, an electric current should give at the place of junction (here vacuum) a heat generation when sent around in one, and a cooling when sent around in the opposite sense.

Suppose that within the enclosure containing the plates (Fig. 23) everything is kept at a constant temperature T by means of a heat reservoir. We have then to calculate how much heat must be supplied or absorbed by the latter when an electric current passes through the system. Let the current flow from the plate 1 to the plate 2. Then the negative electrons move from 2 to 1. The heat taken up by the reservoir during a given time will be equal to the energy of the electrons which during this time enter into the system at P less the energy of those

which, during the same time, leave the system at Q, both the kinetic and the potential energy being taken into account.

Since the temperature is the same at P and Q, there will be no difference in kinetic energy. Let the potential of the plate 1 be higher than that of the plate 2, and let their potential difference Φ differ but little from its equilibrium value. Consider a quantity of electricity -1 streaming from P towards Q. Owing to the potential difference of the plates, this negative charge has at the entrance P into the system a greater potential energy than at the exit Q, the difference being just Φ. In addition to this we have also to take account of the potential energy of the electrons due to the attraction by the metal atoms. The state being stationary, the number of electrons contained within the atoms will remain unaltered. Thus we can imagine that the electrons [involved in the current] pass between the atoms. If the potential energy of an electron in the vacuum is higher by $-ep_1$ than that of an electron in the plate 1 outside the atoms, and by $-ep_2$ higher than that of an electron in the plate 2 outside the atoms, then the negative unit of charge loses in passing from P to Q the amount of energy $p_1 - p_2$. [This is to be added to Φ.]

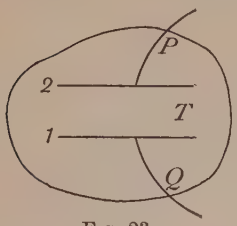

FIG. 23.

Thus the total amount of energy lost by the electrons and, therefore, gained in the form of heat by the reservoir is, for each unit of positive charge passing from 1 to 2, or of negative charge passing from 2 to 1,

$$\Phi + p_1 - p_2.$$

We find, therefore, for the Peltier effect π,

$$\pi = \Phi + p_1 - p_2.$$

37. RICHARDSON'S MEASUREMENTS

The heat generated in such a system has been measured by Richardson.*

In a vacuum was placed a cold plate P_0, kept at a temperature T_0, and opposite it a hot plate P, whose constant temperature T

* *Phil. Mag.* (6), xx., 1910, p. 173.

was so high as to make it emit electrons. In order that all these may reach the plate P_0, the potential of the latter was kept an amount Φ above that of P.

Under these conditions Richardson measured the heat generated on the plate P_0.

We consider the plate P_0 together with the vacuum next to P and a part of the wire attached to P_0 (and consisting of the same metal as P_0) as a system whose gain in energy has to be determined. This will be due, in the first place, to the kinetic energy of the electrons entering the system. In order to determine this energy we require the mean squared velocity of the electrons flying into the vacuum. Now, the number of electrons emitted per unit time and unit area by the plate P, with velocities contained between v and $v+dv$, is *

$$\pi\, ae^{-hv^2}v^3dv,$$

where a is a constant. Consequently, the mean squared velocity with which the electrons enter the vacuum,

$$\frac{\int_0^\infty \pi a e^{-hv^2} v^5 dv}{\int_0^\infty \pi a e^{-hv^2} v^3 dv} = \frac{\pi a/h^3}{\pi a/2h^2} = \frac{2}{h} = \frac{4}{3}u^2, \dagger$$

where u^2 is the mean squared velocity of the electrons in the metal. The mean velocity of the electrons is thus greater at their arrival in the vacuum than within the metal, which is to be explained by the circumstance that only the electrons endowed with the greatest velocities leave the metal.‡

By what precedes, in conjunction with $m/2h = kT$, the kinetic energy entering the system per unit time can be written $2nkT$, where n is the number of electrons passing per unit time from P to P_0. As against this, the kinetic energy of the electrons escaping through the wire amounts to $\frac{3}{2}nkT_0$.

Now for the potential energy of the electrons. In the first place, owing to the potential difference Φ of the plates the energy of the electrons entering the system exceeds by $-en\Phi$ that of

* Cf. p. 176, (61), after integration over θ. † Cf. p. 137.

‡ This is disregarded by Richardson. The final result, however, remains unaffected, since the difference in kinetic energy between the electrons moving in and out is, after all, negligible when compared with the difference in potential energy.

the electrons leaving the system (e being always negative). In the second place, owing to the attraction by the metal atoms, the potential energy of an electron will be greater outside than within the system. Denoting, therefore, this excess by $-e\psi$ we shall have for the heat generated on the plate P_0

$$W = n(2kT - \tfrac{3}{2}kT_0 - e\Phi - e\psi).$$

In Richardson's experiments $e\Phi + e\psi$ was so much greater than $2kT - \frac{3}{2}kT_0$, that we may as well write

$$W = -n(e\Phi + e\psi).$$

The plate P_0 was inserted in one of the branches of a Wheatstone bridge, so that the heat generated could be measured through the change of the resistance. Richardson made use of the circumstance that W is a linear function of Φ. Taking his measurements at different values of Φ and determining the ratio of the W's, he was able to calculate ψ.

This quantity appeared to be different according to the manner in which the piece of platinum was treated before the experiment, and depends probably on the presence or absence of occluded gases. The value found for ψ by Richardson was again 4 to 5 volts, and thus agreed with the value found by previous measurements (cf. Art. 31).

There remains, however, yet one great difficulty. We saw (Art. 33) that the potential difference of 5·5 volts which was obtained in Richardson's experiments described in Art. 31 had to be explained mainly by the difference in potential energy of the electrons in the vacuum and of the intra-atomic ones, viz. by assuming that the electrons are strongly bound to the atoms. Now, in connection with the last experiments there is no question of electrons in the atoms, so that the potential difference of 4 to 5 volts must here be explained by the difference in potential energy of the electrons contained in the metal but between the atoms and the electrons in the vacuum. Thus the agreement of the results of the two sets of measurements is rather unintelligible, and it must be admitted that not all is clear in these investigations.

CHAPTER VIII

PROBLEMS IN WHICH THE MOTION OF ELECTRONS PLAYS A PART

38. Nichols' Experiment

The presence of free electrons in a metal suggested to Nichols * the following experiment. He set a metal disc into rapid spinning motion about its axis, expecting the electrons to be driven by the centrifugal force towards the rim and thus to produce a potential difference between the rim and the centre of the disc. Let $-e$ be the charge and m the mass of an electron, ω the angular velocity of the disc, and Φ the potential at a point of the disc at a distance r from its centre. Then the condition of equilibrium is

$$m\omega^2 r = -e\frac{d\Phi}{dr},$$

whence, by integration,

$$\Phi = -\frac{1}{2}\frac{m}{e}\omega^2 r^2 + \text{const.},$$

or, if Φ_1 be the potential at the centre and Φ_2 that at the rim (R being the radius of the disc),

$$\Phi_1 - \Phi_2 = \frac{1}{2}\frac{m}{e}\omega^2 R^2.$$

The question is whether this potential difference can be detected. If we take

$$\omega = 100 \times 2\pi \text{ sec.}^{-1},$$

$$\frac{e}{m} = 1{\cdot}8 \,.\, 10^7 \text{ (electromagnetic units)},$$

$$R = 10 \text{ cm.},$$

then

$$\Phi_1 - \Phi_2 = 1{\cdot}1 \text{ electromagnetic units} = 1{\cdot}1 \,.\, 10^{-8} \text{ volt.}$$

* *Phys. Zeitschrift*, vii., 1906, p. 640.

Nichols spent much time in trying to detect this potential difference, but his apparatus seems not to have been sensitive enough. Owing to the rotation of the disc, the measurement of the potential difference was connected with great difficulties; sliding contacts had to be applied. He arrived at the conclusion that a potential difference 1000 times as great, such as should manifest itself if the positive ions were free to move, could be detected by his apparatus. The absence of the effect was thus a proof that the positive ions are bound.*

39. Kinetic Energy of the Conduction-current Electrons

The following question suggests itself in connection with the preceding considerations. Can the magnetic energy $\frac{1}{2}Li^2$ of an electric current, which is often referred to as the [electro-] kinetic energy, be interpreted as the total kinetic energy of the electrons, $\Sigma(\frac{1}{2}mv^2)$? The answer is decidedly in the negative. For $\frac{1}{2}Li^2$ depends on the self-induction and, therefore, on the shape of the wire, and is in common cases much greater than $\Sigma(\frac{1}{2}mv^2)$.

The kinetic energy of an electron, whether entirely or partly of electromagnetic nature, is at any rate localised in the immediate neighbourhood of the electron. The total kinetic energy of all electrons is found by addition, since the spaces in which this energy is located do not overlap. This gives $\Sigma(\frac{1}{2}mv^2)$. On the other hand, the quantity $\frac{1}{2}Li^2$ represents the total magnetic energy to which every electron contributes through its weak magnetic field which extends to a considerable distance. In the case of an electric current, however, the magnetic fields of different electrons are equally directed and will, surely, overlap. Since the magnetic energy is a quadratic function of the magnetic force, the total magnetic energy will not be obtained by simply adding up the energies due to the separate electrons. Thus, for instance, if there are N electrons and if each of them produces at a given point of space the same magnetic force H of exactly the same direction, the resultant magnetic force will be NH, and the

* [This experiment has since been carried out successfully, in a modified form and with more refined means, by R. C Tolman and others. For its recent history and the results obtained see *Physical Review*, vol. viii., 1916, p. 97 and p. 753; vol. ix., 1917, p. 164; vol. xxi., 1923, p. 525; vol. xxii., 1923, p. 207.]

magnetic energy will be proportional to N^2, while the quantity $\Sigma(\frac{1}{2}mv^2)$ is proportional to N itself.

Since the number of electrons involved in an electric current is very large, it will be seen that $\frac{1}{2}Li^2$ may be much greater than $\Sigma(\frac{1}{2}mv^2)$.

For the sake of illustration let us consider a simple case in which the conductor consists of a cylindrical wire sheathed by a co-axial tube. Let the current i flow upwards in the wire and downwards in the sheath. The magnetic force at a distance r from the axis is

$$H = \frac{i}{2\pi cr},$$

where c is a constant, the propagation velocity of light. If a_1 and a_2 be the radii of the wire and the tube, the magnetic energy in the space between them is, per unit length,

$$\frac{i^2}{4\pi c^2} \log \frac{a_2}{a_1} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

On the other hand, if N_1 and N_2 be the numbers of the electrons, per unit length, in the wire and the tube, respectively, and v_1, v_2 their velocities, the kinetic energy of the electrons will be

$$\tfrac{1}{2}m(N_1v_1^2 + N_2v_2^2). \quad . \quad . \quad . \quad . \quad (71)$$

Let us now compare the expressions (70) and (71).

We have $i = N_1ev_1 = N_2ev_2$ and, using the electromagnetic mass, $m = \dfrac{e^2}{6\pi Rc^2}$ (if R be the radius of an electron). Thus (71) becomes

$$\frac{i^2}{12\pi Rc^2}\left(\frac{1}{N_1} + \frac{1}{N_2}\right).$$

Now, if a_2/a_1 is a moderate number, (71) will be much smaller than (70), provided that

$$\frac{1}{3R}\left(\frac{1}{N_1} + \frac{1}{N_2}\right).$$

is a very small fraction.

For that purpose RN_1 and RN_2 must be very large, that is to say, the number of free electrons in the conductor, contained between two parallel planes at a distance equal to the radius of the electron, must be very great.

Take $R = 1{\cdot}5 \,.\, 10^{-13}$ cm. and consider a copper wire of 1 cm.^{2} cross-section. Then the number of centres of metal atoms contained between the said planes will be $14 \,.\, 10^9$.

This number is so great that the number of free electrons can satisfy the requirement of making RN very large, and yet be small compared with the number of atoms,—the latter condition being indispensable in order to avoid difficulties with regard to specific heat.

Only for extremely thin wires (of a diameter of the order of a wave-length of light) would the value of $\frac{1}{2}Li^2$ be comparable with that of $\Sigma(\frac{1}{2}mv^2)$.

SOURCES AND REFERENCES

Chapter I

H. Helmholtz und G. v. Piotrowski, *Über Reibung tropfbarer Flüssigkeiten*, Wiener Sitzungsber. xl., Abt. I., 1860, p. 607.

A. Einstein, *Zur Theorie der Brown'schen Bewegung*, Ann. der Phys. xix., 1906, p. 371.

Chapter II

J. C. Maxwell, *On the Viscosity or Internal Friction of Air and other Gases*, Phil. Trans. clvi., 1866, p. 249.

A. Kundt und E. Warburg, *Über Reibung und Wärmeleitung verdünnter Gase*, Ann. Phys. u. Chemie, clv., 1875, pp. 337 and 525.

Chapter III

M. Knudsen, *Die Gesetze der Molekularströmung und der inneren Reibungsströmung der Gase durch Röhren*, Ann. der Phys. xxviii., 1909, p. 75.

—— *Die Molekularströmung der Gase durch Öffnungen und die Effusion*, Ann. der Phys. xxviii., 1909, p. 999.

—— *Eine Revision der Gleichgewichtsbedingung der Gase. Thermische Molekularströmung*, Ann. der Phys. xxxi., 1910, p. 205.

—— *Thermischer Molekulardruck der Gase in Röhren und porösen Körpern*, Ann. der Phys. xxxi., 1910, p. 633.

—— *Ein absolutes Manometer*, ibid. xxxii., 1910, p. 809.

—— *Eine Methode zur Bestimmung des Molekulargewichts sehr kleiner Gas- oder Dampfmengen*, ibid. xliv., 1914, p. 525.

J. E. Shrader and R. G. Sherwood, *Production and Measurement of High Vacua*, Phys. Rev. (2), xii., 1918, p. 70.

M. Knudsen, *Die molekulare Wärmeleitung der Gase und der Akkomodationskoeffizient*, Ann. der Phys. xxxiv., 1911, p. 593.

M. v. Smoluchowski, *Über Wärmeleitung in verdünnten Gasen*, Wied. Ann. lxiv., 1898, p. 101.

—— *Über den Temperatursprung bei Wärmeleitung in Gasen*, Wiener Sitzungsber. cvii. Abt. IIa, 1898, p. 304, and cviii. Abt. IIa, 1899, p. 5.

P. Lasareff, *Über den Temperatursprung an der Grenze zwischen Metall und Gas*, Ann. der Phys. xxxvii., 1912, p. 233.

Chapter IV

G. L. Lesage, *Loi qui comprend toutes les attractions et répulsions*, Journal des Savants, 1764.

A. Einstein und L. Hopf, *Statistische Untersuchung der Bewegung eines Resonators in einem Strahlungsfeld*, Ann. der Phys. xxxiii., 1910, p. 1105.
H. A. Lorentz, *Rapport sur l'application au rayonnement du théorème de l'équipartition de l'énergie. Cas particulier d'un électron libre*, Rapp. Réunion de Bruxelles, 1911, p. 37.
A. D. Fokker, *Sur les mouvements browniens dans le champ du rayonnement noir*, Arch. néerl. (IIIa), iv., 1918, p. 379.

Chapter V

G. Stokes, *Mathematical and Physical Papers*, Cambridge, 1880, i. p. 100.
Rayleigh, *The Theory of Sound*, London, 1896, ii. p. 312.
G. Kirchhoff, *Über den Einfluss der Wärmeleitung in einem Gase auf die Schallbewegung*, Pogg. Ann. cxxxiv., 1868, p. 177; Ges. Abh., Leipzig, 1882, p. 540.
H. Kayser, *Bestimmung der specifischen Wärme für Luft bei constantem Druck und constantem Volumen durch Schallgeschwindigkeit*, Wied. Ann. ii., 1877, p. 218.
P. Lebedew, *Die Grenzwerte der kürzesten akustischen Wellen*, Ann. der Phys. xxxv., 1911, p. 171.
N. Neklepajev, *Über die Absorption kurzer akustischer Wellen in der Luft*, Ann. der Phys. xxxv., 1911, p. 175.
J. H. Jeans, *The Dynamical Theory of Gases*, Cambridge, 1916, second edition, p. 374.
H. A. Lorentz, *Les équations du mouvement des gaz et la propagation du son suivant la théorie cinétique des gaz*, Arch. néerl. xvi., 1881, p. 1.

Chapter VI

O. W. Richardson and F. C. Brown, *The Kinetic Energy of the Negative Electrons emitted by Hot Bodies*, Phil. Mag. (6), xvi., 1908, p. 353.
O. W. Richardson, *The Kinetic Energy of Ions emitted by Hot Bodies*, Phil. Mag. (6), xvi., 1908, p. 890, and xviii., 1909, p. 681.
—— *The Electrical Conductivity imparted to a Vacuum by Hot Conductors*, Phil. Trans. A, cci., 1903, p. 497.
—— *The Emission of Electricity from Hot Bodies*, London, 1916.

Chapter VII

O. W. Richardson and H. L. Cooke, *The Heat developed during the Absorption of Electrons by Platinum*, Phil. Mag. (6), xx., 1910, p. 173

Chapter VIII

E. F. Nichols, *Die Möglichkeit einer durch zentrifugale Beschleunigung erzeugten elektromotorischen Kraft*, Phys. Zeitschr. vii., 1906, p. 640.
A. Schuster, *On Electric Inertia and the Inertia of Electric Convection*, Phil. Mag. (6), i., 1901, p. 227.
H. A. Lorentz, *The Theory of Electrons*, Leipzig, 1909, p. 47.

Further developments of Richardson's thermionic theory were made, among others, by I. Langmuir, W. Schottky, J. A. Fleming:

I. Langmuir, *The Effect of Space Charge and Residual Gases on Thermionic Currents in High Vacuum*, Phys. Rev. (2), ii., 1913, p. 450.

—— *Thermionenströme im hohen Vakuum.* I. *Wirkung der Raumladung.* II. *Die Elektronenemission seitens des Wolframs und die Wirkung von Gasresten,* Phys. Zeitschr. xv., 1914, pp. 348, 516.

W. Schottky, *Die Wirkung der Raumladung auf Thermionenströme im hohen Vakuum,* Phys. Zeitschr. xv., 1914, p. 526.

—— *Über den Einfluss von Potentialschwellen auf den Stromübergang zwischen einem Glühdraht und einem koaxialen Zylinder,* Phys. Zeitschr. xv., 1914, p. 624.

—— *Über Raumladungswirkungen bei Strömen positiver Ionen im hohen Vakuum,* ibidem, p. 656.

—— *Über den Einfluss von Strukturwirkungen, besonders der Thomson'schen Bildkraft, auf die Elektronenemission der Metalle,* ibidem, p. 872.

J. A. Fleming, *The Thermionic Valve and its Developments in Radiotelegraphy and Telephony,* London, 1919.

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